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Re-observing the emergence of mathematics learning through conversations in a classroom from an enactivist perspective
A methodological study

Re-observing the emergence of mathematics learning through conversations in a classroom from an enactivist perspective: A methodological study

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Author’s declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: .... DATE: ....

One becomes a painter by painting.

Vincent van Gogh, 2017, p. 45

Abstract

From my observer perspective, how can I learn from interactions in a mathematics classroom? In this thesis, I am focusing on how my own learning began to evolve through the observation of mathematics knowing emerging from the details of the interactions during conversations in a mathematics classroom both between the teacher and students and amongst the students themselves. The contribution to knowledge is therefore methodological.

I bring an enactivist approach to the study, which took place in a Chilean grade-eight classroom (ages 13–14 years old students), when they were doing mathematics in their usual way and also working with a mathematical modelling task for the first time.

My methodological stance to this study is also enactivist, allowing me to carry out different observations as the mathematics knowledge emerges through the interaction of the participants. As a result, I conducted my work during recursive cycles of observation and re-observation of data. The first observation implies an explicit and descriptive categorisation of the mathematical action of the participants delivered by methods such as observations, video- and audio-recorded lessons and interviews. The second observation focuses on mathematical episodes, detailed and selected mathematics conversations recorded by re-observation of the participants.

I discuss the role of the 'second' observer in the use of video-recordings when the observations of the classroom were carried out by the same observer, drawing the observations under two types of categorisation: basic and subordinate.

I carried out my analysis of the data using episodes of mathematics, based on my observations of the conversations noting how my own learning emerges through the interactions that I have with this study. From these observations, as a theoretical framework, I bring forth emotions, the process of becoming aware and actions of distinction. I also considered the levels of categorisation of my observations.

This analysis uncovered the following characterisation of learning when students and teacher are involved in solving a mathematics problem. They perform actions to become coherently aware mathematically in their classroom, noting moments of: suspension, redirection and letting go (Depraz, Varela, & Vermersch 2000, 2003; Varela, 2000). There is an empathy between student and teacher and amongst the students linked to an interpretation of what they are doing mathematically. The distinctions made in the micro-historicity of each teacher or student can trigger similar interactions in the mathematics classroom.

As observer, I argue for a characterisation of the learning process to include the micro-historicity of each participant. Such an approach encourages me to observe the details and the shifts in action, which lead to considering the process of actions through which a sense of empathy can lead to an understanding or misunderstanding of a mathematical situation. By considering the distinctions of each participant, I am able to see the similarities and mathematical influences in the interactions between the teacher and their students, and the students themselves. The desired mathematical awareness begins for the student in the process of interactions with others. To observe this, it is necessary to follow the particular interactions of each participant noting the underlying awareness.

Abstract (Spanish version)

Siendo observador, ¿Qué puedo aprender de las interacciones que se dan dentro de una clase de matemáticas? En esta tesis, me estoy enfocando sobre como mi propio aprendizaje comienza a surgir por medio de la observación del conocimiento matemático que emerge desde los detalles de la interacción durante conversaciones entre el profesor y sus estudiantes y también entre los mismos estudiantes en una sala de clases. La contribución al conocimiento por lo tanto es metodológica.

He adoptado una posicion enactivista para llevar a cabo mi estudio, el cuál se ha desarrollado en una clase de 8 Básico de matemáticas en Chile (con estudiantes entre 13 y 14 años) cuando estaban trabajando matemáticas en su forma común como también resolviendo por primera vez una actividad de modelamiento matemático.

La posición metodológica que he usado en este estudio es también enactivista, permitiéndome llevar acabo diferentes observaciones cuando el conocimiento matemático surge a través de la interación de los participantes. Como resultado, he dirigido mi trabajo en ciclos recursivos de observacion y re-observacion de los datos. La primera observación implicó una categorización explicita y descriptiva de las acciones matemáticas realizadas por los participantes, recopiladas usando métodos tales como: observacion, video-audio grabado de la clase y entrevistas. La segunda fase de observación se enfocó en detalladas y seleccionadas conversaciones, las cuáles nombro como episodios y que provienen de la re-observación de los datos recopilados sobre los participantes.

He discutido el rol de ‘segundo’ observador en el uso de videos grabados cuando se da que el observador de estos, es el mismo que realizó la observación en primera instancia en la sala de clases; Estas observaciones son realizadas usando dos tipos de categorias: básica y subordinada.

Basado en mis observaciones de las conversaciones sobre matemática, realizo mi análisis considerando episodios de matemáticas, notando como mi aprendizaje surge a través de la interacción que tengo con este estudio. Desde estas observaciones, como marco teórico nace emociones, procesos de conciencia y acciones de distinguir. Además, también considero los tipos de categorías de mis observaciones.

Cuando el profesor de matemáticas y sus estudiantes están envueltos en resolver un problema matemático, mi análisis descubre la siguiente caracterización del aprendizaje: Los participantes realizan acciones de coherente conciencia matemática dentro de su clase notando momentos de, suspensión, redireccionar y dejarlo ir (Depraz, Varela, & Vermersch 2000, 2003; Varela, 2000). Existe una empatía entre estudiantes y profesor, y entre los estudiantes que está asociada a la interpretación de lo que están haciendo matemáticamente. En la clase de matemáticas, la distinción realizada de cada participante (profesor y estudiante) pueden desencadenar similares acciones, las que son distinguidas a través de la micro- historicidad.

Como observador, argumento que una caracterización del proceso de aprendizaje, incluír la micro-historicidad de cada participante, como un acercamiento que me promovió observar los detalles y los cambios dentro de la acción. Este acercamiento lideró en considerar la empatía dentro de los procesos de acción, la cuál puede lidear con entendimiento o no de una situación que envuelve matemática.

Además, bajo el mismo acercamiento, tomando en cuenta la distinciones de cada participante, yo pude observar las similitudes e influencias matemática dentro de la interacción que se da entre el profesor y sus estudiantes, como entre los mismos estudiantes. La conciencia matemática en el estudiante comienza en el proceso de interacción con otros, sin embargo, para observar esta situacion, es necesario dividir la interacción observando la conciencia subyacente de cada participante.

To my beloved baby Benjamin and my wonderful husband Rodolfo. Thanks for all your love. I owe you so much.

Acknowledgments

Completing this work has been a tremendous and exciting way of learning. Thanks a lot for those wonderful people that I have met within almost four years here.

I would like to thank the mathematics teacher, their students and the school who provided me with the access of being with them and learning from their doing within this dissertation. It has been a pleasure being there.

I would like to acknowledge CONICYT for the scholarship from Chile for providing the funding for doing this PhD. Thanks to Santander for the travel grant, supporting my fieldwork in Chile and also thanks to the Alumni Foundation and School of Education from the University of Bristol for supporting my attendance and presentations at international conferences and the summer school where I learnt from so many interesting researchers.

Thank you to my dear supervisors Laurinda Brown and Rosamund Sutherland. Laurinda, doing this PhD has been truly a joy; thanks for all your dedication, insightful comments, kindness and freedom with my work. Ros, a long time ago, somebody asked you, “Who is she?”, and you replied, “She is my last dance”. I hope you can see how this dance is finishing in some place of heaven.

I would like to thank my family for all their love for us. I must express my gratitude to my little baby Benjamin (7 months by the submission of this thesis) and also my dear husband, Rodolfo. You do not know how much I have learnt from both of you and how much I love you; thanks for your unconditional time and love.

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Thesis outline

This thesis came into being in the form of a path of my own learning as a researcher, occurring as one of the consequences of my awareness of learning about mathematics.

I have decided to write this thesis in the form of a dynamic spiral (see figure 1).

Figure 1: Photograph of spiral taken by David Reid.[1]

This means that the chapters are interconnected; sometimes the reader can note a 'back and forth' movement between them, with references to other chapters written previously or later.

However, in order to make an outline of this work and perhaps facilitate the reading and the reader's own pathway, I have divided this thesis into three parts, with each one composed of chapters. Part one consists of my experiences, concerns and dilemmas as a result of being involved in mathematics education, showing how I have come to work with my epistemological position enactivist theory, highlighting some concepts of the theory. Part two refers to learning, my methodological position enactivism, interaction and observation, presenting a theoretical framework that I later use as a part of my view of learning mathematics. Part three relates to my empirical

[1] I had a photograph of a dynamic spiral in this position but, unfortunately, during final checks, the picture is now not available on the web. My supervisor, Laurinda, gave me this this one as a placeholder.

work in a Chilean mathematics classroom, including my research findings. If, for instance, as a reader, you would like to read about the research project first, I suggest you begin with part 3. For the research project, I observed a mathematics teacher and 23 students (ages 13–14 years old), when they were working on mathematics in their usual way to solve word problems and on exponents, powers, square roots, percentages and also doing a mathematical modelling task for the first time.

Finally, inspired by the idea of a metaobserver, whose role is “to provide feedback to the observer and also learns observation from observing an observer” (Cronholm, Guss, & Bruno, 2006, p. 7), I decided to introduce a metaobservation at the end of each chapter. To me, a metaobservation is a re-observation of what I have described in each chapter, adopting a reflective tone about what I have learned, for the purpose of giving the reader the opportunity to follow my learning through this study.

PART ONE: A story of experience

CHAPTER 1: Introduction

1.0 Introduction

This chapter presents an account of how I started thinking about mathematics learning, describing my changes as a researcher when I was moving to my first researcher questions of this doctoral study.

To give this account, I start with a section named *My journey*, which explains how different actions in my life led me to think about learning in mathematics. I note how my ways to be in the world and the interactions I had as a young learner (in school), mathematics teacher and researcher influenced my thoughts. The second section shows how I started to move onto my research questions, noting my interest in mathematical modelling.

I end this chapter with a metaobservation, looking back and seeing how 'real life' problems start to resonate with my thinking and noting the inseparability between a mathematics world and me.

1.1 My journey

> Where we came from and where we are going.
> (Maturana and Varela, 1992, p. 27)

I would like to start with a personal story about mathematics. Although I can remember neither when nor how I first learned mathematics, it became a major part of my life at a young age. I began as a child by working alone at my desk. Later, I used my home as a regular study spot with classmates to work on algebra and calculus in high school. Even during my undergraduate years, studying with classmates and friends in the mathematics group was part of my everyday life, although I also spent solitary time working on mathematics problems. Working on hard mathematical puzzles meant that we were challenged within the group and got different things wrong. Discussing, when together, supported us in gaining solutions.

Within my undergraduate studies, centred in pure mathematics, one of the usual actions was proving, solving mathematics properties, such as the triangle inequality for real numbers, based on, for instance, theorems and lemmas. In this context, perhaps influenced by my experience as a teaching assistant in different mathematics courses and my personal history with mathematics, I realised that there is not one way to arrive at the final solution of a mathematics problem; rather, the solution depends on the solver's understanding of the world and the mathematical context in which he or she is working.

My first investigation into this area was a requirement for my bachelor's degree in education and the teaching of mathematics. I developed a small study, based on the Theory of Registers of Semiotic Representation (TRSR; Duval, 1995; 2006). This theory briefly says that there are many mathematics representations (i.e., cartesian graphs, written language, written symbols) to observe the same mathematics object. A person, either student or teacher, who is involved in solving a mathematics problem can use these different representations. For example, in the following mathematics word problem:

> Tom received £12 to feed a neighbour's cat for 3 days. At this pay rate, how many days will he have to feed the cat to earn £40?[2]

Reading the mathematics word problem above, the variable *days*, which is in written language, can also be written as a symbol, *d*, where *d* represents Tom's working days. The variables, *days* and *d,* are in different registers. In this context, there is a change of representation for the same situation, the *days* in the word problem. I used this idea to study the transfer of written language to algebraic language (written symbols) when a teacher and a group of students not taught by them were solving linear programming problems.

One aspect of linear programming tasks is that they work with different variables to distribute resources in an optimal way. For example, a shop sells two types of tea, A and B. The manager pays £4 for each unit of tea A and £7 for each unit of tea B. The profit for each unit of A and B, respectively, is £1 and £1.50. The manager of the store estimates that if he invests no more than £10,000 on teas A and B, he will sell no more than £1000 worth of teas each month. How many units of each of teas A and B should be stocked in order to maximise the profit? The variables are the number of units of each type of tea, A and B. The function $P(A,B) = A + 1.5B$ is to be maximised according to the constraints in the above problem.

In addition, the mathematical interpretation of the situation can be modelled through the use of graphs, functions and numbers. In the teas example, there is a

[2] Problem from Open-Ending questions for mathematics, p.10 Dr. Ron Perfrey, Mathematics Consultant, online at https://www.uky.edu/OtherOrgs/ARSI/www.uky.edu/pub/arsi/openresponsequestions/grade4orq.pdf , accessed 9th July, 2019.

transfer between written language, “The profit for each unit of A and B, respectively, is £1 and £1.50” and algebraic written symbols, P(A,B) = A + 1.5B.

In this work, I wanted to understand how the students and the teacher worked with linear programming word problems to make mathematics ‘more real’ by playing with different variables until finding a solution.

Later, drawing upon my undergraduate studies as a mathematics teacher, I attempted to apply pure mathematics to common situations. For example, at the beginning of a number theory lecture in 2003, I learnt about congruence modulo n. I was using a different counting method, for instance, working in the context of modulo 5, counting will simply be 0, 1, 2, 3, 4. I used this same idea for counting and sometimes asked my students the following, “Why can we say that thirteen hundred hours is 1:00 pm?” Different answers emerged, with the primary one being, “Because of twelve”. I asked what that meant, “Why twelve?” When teaching, I would always finish telling a story about counting by reminding students to consider modulos. This idea of mathematics connected to their daily life, as a way of making sense of what we know mathematically, is still important to me.

From 2007–2009, while working as a mathematics teacher, I participated in a small research group that included mathematics teachers from different schools. The main researcher was from my home university. The goal of the group was to create a mathematics situation for students using a calculator that projected their answers onto the whiteboard.

The activity designed worked well. Students were excited about the use of technology. They immediately saw the answers on the whiteboard and compared them to their own. This was not a planned reaction but rather one that occurred naturally. My dilemma was how to construct an activity that offers a clear representation of mathematics. I wanted to connect mathematics with their lives again. One issue that began to catch my attention was unexpected ideas from my students.

The classroom surroundings and the experiences lived in it became increasingly important in my teaching life. Within this period, I worked in three different environments: teaching mathematics in a high school (with excellent outcomes at the national level); working with undergraduate students who had arrived with lower mathematics knowledge (according to the statistics of the higher education technical college, which is a place of learning that is dedicated to creating professionals without a bachelor's degree); and working in policy as part of the national mathematics curriculum, which included a variety of tasks, such as developing resources to support teachers and students with the new mathematics curriculum and an in-depth review of the status of mathematics in an international context.

At this time, I was also working on my master's degree, attempting to understand how students reflect when they work on activities related to mathematical modelling problems. This was another effort to connect daily life thinking to mathematics.

Mathematical modelling had become of particular interest to me because it offers ways to perceive the world via the experience that we have with it and to learn from this experience to set up another mathematical model. Working with mathematical modelling tasks provides an open situation with mathematical variables in which the final outcome (the model) might not be the same for each student but needs to be explained mathematically. For example, by using a model, I can find the profits of a business once I know the cost and sales of a product.

At different periods of time (2009-2015), I worked in two high schools (students aged 11-18 years old) in which, as a matter of school policy, we were asked to voluntarily observe the mathematics lessons of other colleagues. This allowed me to reflect on my own teaching practices and to note that students' and teachers' behaviours, when doing similar mathematics, often differ.

These contrasts in environment, and my observations about variability in teaching and learning mathematics, allowed me to see that each person is unique and will develop their own relationship with mathematics over time. Therefore, one needs not only to consider a situation described on paper and the resources, technology, teacher and students; but also, mathematical thoughts, background and stories of each student and the teacher. Their interactions *in situ* are also critical factors.

At the same time (2010-2015), while I was working as a mathematics teacher in a technical college, I implemented a new way of teaching, for me, on some courses in which I worked with the students instead of merely speaking at them. To do so, our chairs were placed in a circle so that we could see one another. One student would read a mathematics problem, followed by another student voicing an opinion about it, which would often remind the others of a similar experience. My role was to pose questions about what each student said and to deliver spontaneous feedback. That feedback generally led to more questions. When everyone had demonstrated their understanding, we would move on to another mathematical problem. This meant that our class was not only concerned with an outcome but also with what we were doing in a particular moment—the interaction *in situ* that was mentioned above.

My students began exhibiting positive changes in the way in which they perceived mathematics. I found myself learning mathematics with my students in becoming a part of this process.

This idea of learning mathematics from the students is also mentioned by Pimm (2019, p. 29) in an article recalling the teaching life of David Henderson, an academic mathematician, who worked,

> [l]istening acutely to students as they worked on mathematics, learning himself about mathematics from what they [the students] said, wrote, drew and did.

Interaction between teacher and students entails opportunities to learn mathematics from other perspectives.

Due to my experiences with the mathematical learning environment, as teacher or learner, the way each person becomes involved in it through action has become a primary interest of mine.

1.2 Moving onto my research questions

When I started the doctoral study, I was going to characterise the emergence of mathematical modelling. For me, at that time, mathematical modelling was a way to perceive the world, turning, for example, written language from a 'real' open situation into symbolic language by setting up a model using the experience from the 'real world' (discussed in detail in chapter 8, section 8.1, pp. 112-118).

My interest in mathematical modelling began with my master's dissertation related to the process of reflection that can lead students (age 15-16 years old) to mathematical modelling skills. In addition, mathematical modelling is a learning goal which has recently been integrated into the Chilean national curriculum (Ministerio de Educación de Chile, 2012c; 2016).

The implementation of mathematical modelling could result in many positive benefits for learning mathematics, "in terms of performance, underlying process, and motivational and affective aspects" (Verschaffel, Van Dooren, Greer, & Mukhopadhyay, 2010, p. 22). However, recent research indicates that teachers are unsure how they should act in this implementation process (Tekin Dede & Bukova Güzel 2016, p. 1). Other research highlights the importance of training for implementing mathematical modelling, because "teachers and their beliefs concerning mathematics must be regarded as essential reasons for the low realization of applications and modelling in mathematics teaching" (Kaiser, 2006, p. 399).

In this context of working with mathematical modelling, Blum and Borromeo (2009, pp. 51-52) suggest some strategies for how teachers should act when they are implementing the processes of mathematical modelling, such as: working independently with teacher support and students working alone; giving the students many opportunities to acquire mathematical competencies; establishing connections within and outside mathematics; varying methods flexibly; using time effectively; and separating learning and assessment.

At the time of writing, it seems that the way each teacher approaches mathematical modelling from their own experience, that is, their actions within mathematics in any implementation of mathematical modelling, has not been reported on. Some research does indicate that teachers do not know how to implement the modelling process. What is happening in the link between mathematics teachers and their environment? How effective are the current training courses for mathematical modelling? There are many factors that can influence these considerations, including: the effectiveness of teacher training; the national curriculum; and whether mathematical modelling is part of the curriculum within a particular academic year.

I decided to start my doctoral research by studying mathematics teachers' beliefs about mathematical modelling (see List of Publications, appendix 1, pp. 270-271), drawing upon a theoretical framework based on beliefs about the learning and teaching of mathematical modelling as well as mathematical modelling itself. My questioning attempted to illuminate what I understood by mathematical modelling and how this could be developed within the context of a lesson and within the environment where teachers and students work.

1.3 Metaobservation

I was influenced by my experiences as a mathematics teacher as well as a learner of mathematics with my students. I was trying to understand my mathematical world, meaning my own experiences in what I was living with others (i.e., students and classmates).

While learning and teaching mathematics, I attempted to understand how students learn mathematics through a situation that is associated with something more common in ‘real life’. Now, I can understand that it is not necessary to look for ‘real-life’ mathematics problems because what I have done or what others have done in mathematics is just experiencing mathematics and that is part of ‘real life’.

Therefore, from this experience of living in the world, there is no division between mathematics and me, because what I am doing is experiencing mathematics, and that is living in the world.

CHAPTER 2: Experiencing mathematics

2.0 Introduction

In chapter one, I presented how I started to learn about learning through my experiences of living. Interaction with others has allowed me to see learning from another perspective. In this chapter, I explore this other perspective, linked to doing mathematics.

In the first section, *Seeing the 'mathematical world'*, I start to observe how paying attention to the interaction with others allows me to see learning from another perspective. In the second section, *Didactical situations*, I explain how I came to see that knowledge and knowing under this theory seem to be divided. The third section explains *a priori* and *a posteriori* analyses of didactical situation. In a fourth section, *Contrasting with my own experience,* I explore my epistemological shift of seeing mathematics learning from a constructivist perspective.

Chapter 2 concludes with a metaobservation, noting how, as a teacher and researcher, I start to leave behind the idea that the student must know i.e., 'must know fractions because they studied them last year' and move to sense of doing mathematics now.

2.1 Seeing the 'mathematical world'

In an attempt to understand my mathematics learning as a novice teacher, in chapter 1 I offered an account of my personal history of learning and teaching mathematics. As an experienced teacher, I was not seeing the world of my classroom centred on theorems, algorithms and proving a mathematical task, which could be seen as a positivist view where there are beliefs “that objective accounts of the world can be given, and that the function of science is to develop explanations in the form of universal laws” (Punch & Oancea, 2014, p. 18). I was highlighting the experientiality in mathematics at the moment of doing when I was sitting learning mathematics with my students in a circle (chapter 1, section 1.1, p. 7).

My growing perceptions about how to learn mathematics started a process of change. I had been firmly placed in the tradition of constructivism from my home university. Constructivism “begins with an emphasis on the constructed world of the knower and the relationship of that world to reality” (Lesh & Doerr, 2003, p. 536), which briefly means “knowledge is actively constructed by the [students], not passively received from the environment (Clements & Battista, 1990, p. 34). This ‘construction’ begins with ‘steps’ that the students can do. They are building mathematical knowledge with those steps. This process is quite opposite to ‘mathematical instruction’.

Phrases like knowledge is constructed, the teacher is a guide; and using didactical situations to design mathematics activities to construct knowledge were part of my studies.

2.2 Didactical situations

When I was a teacher, I began participating in a research group in which the members used didactical situations based on a theory from French mathematics education, called *Didactiques des Mathematiques*. The French Didactics of Mathematics in general focuses its studies on working with teachers, students and the mathematical knowledge. It could be illustrated through the didactic triangle (as shown in figure 2) because it is "the classical *trivium* used to conceptualize teaching and learning in mathematics classrooms" (Goodchild & Sriraman, 2012, p. 581).

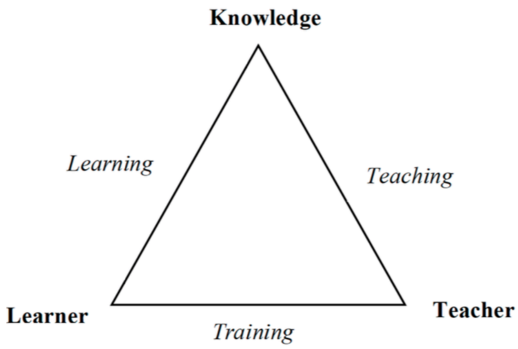


Figure 2: The didactic triangle of Houssaye (1988).

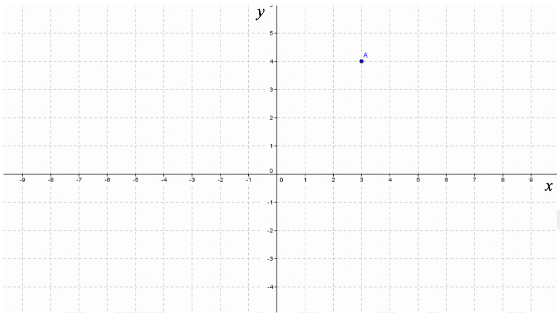
Within the didactics of mathematics and bearing in mind this is a scientific discipline (Bielhler, Scholz, Strässer, & Winkelmann, 2002) there are many branches, such as:

- *didactic transposition*, which means the transformations of mathematical knowledge experimented with in different contexts, since being 'born' until it is taught (Chevallard, 1982).

- *didactic contract*, which is the study of the relationships and expectations of the persons who are involved in a didactical situation, for example, a teacher and their students (Brousseau, Sarrazy & Novotná, 2014).

- *the development of didactical situations*, teachers creating learning activities with the goal of supporting their students in constructing mathematical knowledge (Brousseau, 1997; Straesser, 2007, p. 165). For example, the goal of the following mathematical activity is constructing knowledge of a graphical representation of equations written as x = a, y = b, where x and y are variables dependent on the numerical values of a and b.

1. Find the coordinates of three points, one in each empty quadrant[3], that are the same distance from the origin O(0, 0) as the point A(3, 4).

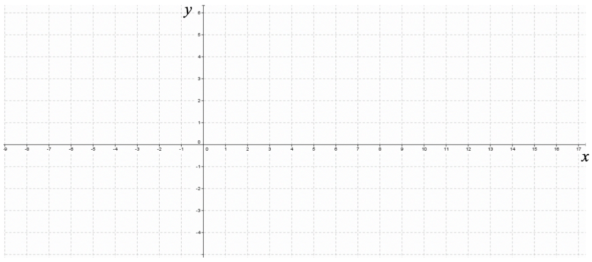


Item 1 refers to looking for and locating coordinates in order to visualise the form of a rectangle if the coordinates are joined. In addition, the activity requires familiarity with the Cartesian coordinate system and the notion of distance. These required actions are part of what in a didactical situation happen first where these actions from the students are "inferred without including the formulations or declarations of validity that can accompany them" (Brousseau, 1997, p. 62). The next item, activity 2, requires

[3] A quadrant refers to each division created by the axes in a coordinate plane. There are four quadrants in the coordinate plane, the first quadrant being to the top right, usually numbered by roman numeral I in Chile, the second quadrant being to the top left, numbered by roman numeral II. The counting goes counter-clockwise until IV.

that the students must create a formulation, writing what they note in mathematical terms when creating a square with sides parallel to the axes (see parts a) and b) below).

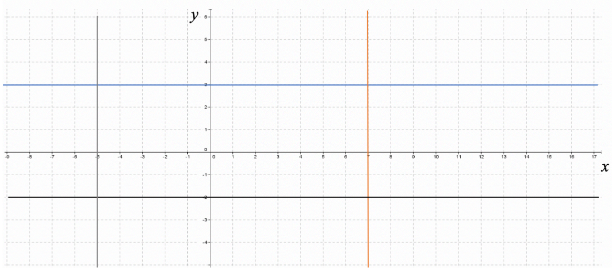
2. On the next pair of axes, create a square with sides parallel to the axes.



a) Observing the coordinates in the vertical lines of the square, what is common between the coordinates?
b) Observing the coordinates in the horizontal lines, what is common between the coordinates?
c) Write formulas to describe the situations in a) and b).

After writing the situation above in mathematical terms according to the instruction to "write formulas", Brousseau (1997, p. 62) proposes a stage of validation in which the students establish, under their own criteria, what mathematical knowledge is for her/him with comparison to what has been done before. Taking account of what would have been done in items 1 and 2, the student in item 3 is able to validate that previous work, as shown the next item:

3. Identify each graph with its equation:



The next item asks for a description of the graph x = a and y = b in general mathematical terms, without plotting them. The move in the activities is from concrete to abstract.

> 4. Consider a and b to be real numbers. Describe a graph in the Cartesian plane for the equations, x = a and y = b.

Brousseau (1997) proposed that a didactical situation, such as the one in the example above, involves the following: (a) *action*, where the teacher and students engage in a specific activity, creating the first approach to the situation. (Items 1 and 2 in the example, plotting the rectangle and square). This first activity involves students in the goal of the situation. (b) *formulation*, such as expressing the situation in mathematical terms, "inferred without discussion of proof" (Brousseau, 1997, p. 62) (In item 2 and in writing formulas, what the student has studied previously will be built upon, supporting the next step, writing the situation in mathematical terms). (c) *validation,* the student needs to verify their assumptions, validating and extending what he/she has done during the previous item in the situation. (In the example, item 4.) Brousseau (1997) says that, "a piece of [mathematics] knowledge is an institutionalized knowing" (p. 62). Generally, institutionalised mathematics knowledge would be expressed by the teacher who is leading the activity in the classroom. (In the example above, this would be the type of line that could be plotted when x = a or y = b). It is hoped that through this process, the student abstracts and fixes the mathematical knowledge behind the activity.

It is possible to infer that, in a didactical situation, knowledge and knowing are separated for the students and the teacher while doing mathematics. Knowledge implies something that must be reached, whereas institutionalised mathematics knowledge is a process designed through a sequence of activities.

2.3 *A priori* and *a posteriori* analyses of didactical situations

In the research group described above (section 2.2), we used components such as an *a priori* analysis of the didactical situation, which comes from didactical engineering (Artigue, 1989) that I had studied in my undergraduate course. This process includes considering what could happen, in mathematical terms, during the lesson.

In the example described in section 2.2, the *a priori* analysis of the didactical situation concerns what could happen, including mistakes, in mathematical terms when the students are involved in the activity, i.e., for item 2, when asked to plot a square the students could draw a rectangle rather than a square (see figure 3).

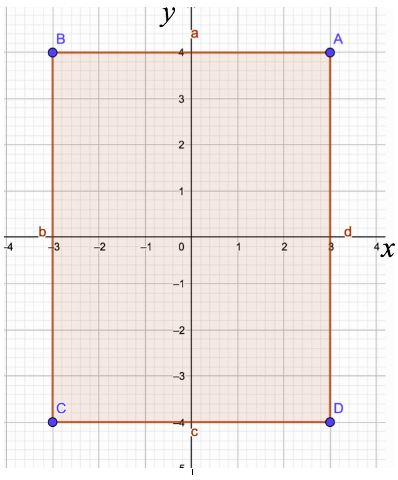


Figure 3: Potential answer by a student drawing a rectangle.

Potential answers, to the questions relating to item 2, will depend on what graph the students have drawn, for example, for the graph described above these would be:

a) Observing the coordinates in the vertical lines of the square, what is common between the coordinates?

- The coordinates are at the same distance from the axes; for example, the shortest distance between the coordinate A and the y-axis is equal to the shortest distance between the y-axis and the coordinate B.
- The common feature between the coordinates in the vertical line AC is that the entire value of coordinate x is equal to 3. Similarly, in the vertical line DB, the entire value of coordinate x is equal to −3.
- The coordinate of x in quadrant I is the opposite of that in quadrant II.

b) Observing the coordinates in the horizontal lines, what is common between the coordinates?

- The coordinates are at the same distance from the axes; for example, the shortest distance between coordinate A and the x-axis is equal to the shortest distance between the y-axis and coordinate D.
- What is common between the coordinates in the horizontal line (AB) is the entire value of coordinate y is equal to 4. Similarly, in the other horizontal line (CD), the entire value of coordinate y is equal to −4.
- The coordinate of y in quadrant III is the opposite of that in quadrant IV.

c) Write formulas to describe the situations in a) and b).

- A formula could be $x = 3$ or $x = -3$.
- The coordinate of x in the quadrant I is the opposite of that in quadrant II.
- The coordinate of y for quadrant I is equal to that in quadrant II, $y = 4$.
- The coordinate of y in quadrant I, $y = 4$, is the opposite of that in quadrant IV, $y = -4$.
- A formula could be $y = 4$ or $y = -4$.

In the examples above, there are more potential answers that could be described. In my opinion, this is because the experience of each student with the task is dependent on how he or she sees mathematics.

In the same research group, after preparing the didactical situation and describing an *a priori* analysis, we continually discussed how the lesson had gone, recording an *a posteriori* analysis. This *a posteriori* analysis came from observation of what the students had done on the task. Through this action, we compared with the *a priori* analysis. One of the consequences of these comparisons was seeing new actions that were observed when the activity was carried out, allowing us to improve our didactical situation and the new *a priori* analysis when it was used again.

2.4 Contrasting with my own experience

The research group and my studies as an undergraduate and at masters levels may suggest that my epistemological position could have been constructivist. This was because I was working with a specific task, for students to construct mathematics and the representation of mathematics, trying to follow what the literature said with the aim of 'provoking' learning.

However, from the theory of didactical situations, I started to note a division between knowing mathematically and mathematics knowledge. Mathematics knowledge is firmly associated with the institutionalisation of what has been studied through the mathematical activity. It is a formalisation of what the students have done (or constructed) previously. In the example of the plotting activity, the didactical situation was representation of equations written as $x = a$, $y = b$ where x and y are variables depending on the numerical value of a and b.

On the other hand, knowing mathematically is linked with the steps by which the students construct their own understanding using previous mathematical concepts they have studied. For example, if the proposed activity, described in the section on didactical situations (chapter 2, section 2.2, pp. 13 -17) was representation of equations written as x = a, y = b where x and y are variables depending on the numerical value of a and b, perhaps the previous concepts needed are plotting coordinates, distance between the coordinates and parallel lines.

Although I could understand what was proposed by didactical situations, what I was living as a teacher was not a division between mathematical knowledge and mathematical knowing. I was doing mathematics and my students were doing mathematics as well.

In addition, reflecting on my own teaching and learning experiences, I began to recognise that the 'anticipation of', planned in the *a priori* analysis, was no more than a reflection of my own experience in mathematics – the distinctions that I had noted when I was studying, learning, teaching, and essentially doing mathematics.

Interestingly, the anticipation of potential answers of the students promoted by *a priori* analysis can be observed in other research in mathematics education. For example, in mathematics misconceptions of the students, Sbaragli & Santi (2011) make a distinction in what the students have done in terms of avoidable and unavoidable misconceptions. Avoidable misconceptions

> *depend precisely on the choices that the teacher makes for carrying out the didactic transposition* and *choices concerning the educational design* (p. 119, emphasis in original)

When the mathematics teacher is carrying out a lesson, in the avoidable misconceptions (given the decisions made by the teacher i.e., didactic transposition and educational design), there is an anticipation of some answers of the students. If we do not know the decisions of the teacher, in line with Sbaragli & Santi (2011), how can the misconceptions of the students be established?

Unavoidable misconceptions are related to "the ontogenetic characteristics tied to the student […] [these misconceptions are dependent] […] on the necessity to say and show something in order to explain a concept" (Sbaragli & Santi, 2011, p. 119).

For me, what is important in any mathematics misconceptions of the students is to acknowledge the participation of students with their surroundings. Their own characteristics that make them unique when doing mathematics, even if the mathematics being done can be seen to be a misconception. For example, in fractions, adding the numerators and the denominators to obtain the addition:

$$\frac{1}{3}+\frac{5}{4}=\frac{6}{7}$$

My attention as a teacher is responding to particular misconceptions rather than creating an anticipation of potential answers of the students.

These experiences of trying to predict students' fixed steps in learning institutionalised knowledge through my own actions as a teacher started to create strong dilemmas between what I was living and observing mathematically with the students.

My questioning allowed me to reflect on what happens during the learning process while considering the surroundings, the context created through the interactions of students and teacher. How the students connect to and make sense of the mathematical task presented are important not only in this research but also in my work as a teacher.

I started to find the unpredictable actions from the students within mathematics, such as unavoidable misconceptions, to be significant.

As a consequence, my thoughts as a mathematics teacher began to be linked more to subjectivity than objectivity when I carried out my teaching activity. I started to find more value in different ways of understanding mathematics in my students. I observed the actions I performed while doing mathematics, and the actions that my students made, each one depending on their own perception or their own 'view' of mathematics. My actions were not fixed in one way of knowing, it was more about doing in mathematics.

2.5 Metaobservation

Although I was trying to understand this way of conceiving knowledge, as constructed, my experiences in doing mathematics and what I lived were different scenarios mentioned in my personal history. This led me to a change of how I see knowing. The actions performed in my history (as a teacher or learner) contrasted and influenced my view of learning, which led me to a sense of doing and, more specifically, a sense of how I taught mathematics. My teaching philosophy changed from thinking that students must learn to a focus on learning mathematics with them.

I began avoiding the idea of pre-knowledge and words such as they (students) 'must know', e.g., they "must know fractions because they studied that last year".

Instead, I started a lesson by asking them to tell me what they know about some mathematics concept, for example, fractions. I let them show me what they knew about fractions in the process of learning mathematics. This openness to uncertain events helped me to pay more attention to what they were doing, trying to note their actions.

I realise that I cannot separate myself from my lived experience, and this can be expressed as “where we came from and where we are going” (Maturana & Varela, 1992, p. 27); therefore, by observing all the changes through which I have lived, I can see that my interest in understanding and learning were always present in some implicit way.

Finally, my understanding of learning shifted from a constructivist position to a sense of doing in mathematics. This was a paradigm shift. I found that my view of learning was more linked with another epistemological position, enactivist theory. Enactivist theory briefly promotes the actions carried out between a person and the environment.

In the next chapter, I will explain more deeply what enactivist theory is to me and how I started to make sense of it within a mathematical context.

CHAPTER 3: Enactivist theory

3.0 Introduction

In the previous chapters, I have explored my own actions as a teacher, learner and researcher and how these experiences led to me to see mathematics learning through the interaction with others. I note my epistemological shift from constructivist to enactivist theory.

In this chapter, I describe that my prior experience influenced how I 'see' the world, moving to a sense of doing. I explore what enactivist theory means to me, noting the central concepts of enactivism that I start to observe in my way to see the mathematics learning, such as, autopoiesis, autonomy, cognition, organisation and structure (including structural coupling and structural determinism) and coherent behaviour. I include a description of enactivism in mathematics education, giving some examples of using this theory.

At the end of the chapter, the metaobservation finishes with my re-observation thinking of "learning to learn" (borrowing some words translated from Spanish from Varela & Flores, 1999), with a question, in particular, "How does the other know what they know?"

3.1 Moving to a sense of doing

My teaching experience and observation opportunities have significantly influenced me and my way of understanding the world has changed. Neither reality nor the task were outside of me and my students. Furthermore, each student interacts with others "to explain or reformulate the observation of a phenomenon" (Maturana *et al*., 1992, p. 28), for example, a mathematics property or a numerical pattern.

To further understand learning in mathematics that is associated with each person and their pathway (their own route of action), story and environment, when I was teaching in the Technical University, I researched and found that León Gómez (2006) and Thompson and Stapleton (2009) share an idea of 'doing' in the environment. This idea originated from the Chilean scientists Maturana and Varela (1992) and their enactivist perspectives: "All doing is knowing and all knowing is doing" (p. 27).

3.2 My relationship with the world in which I am living

> Every act of knowing brings forth a world.
> (Maturana & Varela, 1992, p. 26)

What is my relationship with the world in which I am living? How can I conceive knowledge, considering what I am doing within it? In my previous chapters, I have detailed how being involved as a mathematics teacher in different educational scenarios around mathematics has influenced my way to 'see' the world.

In my way to 'see' the world, I notice the uniqueness of my students through the experience of the behaviour from each one, mathematics knowledge born in each interaction. Naturally, this is not static knowledge, it is changeable and not fixed. Knowledge, for me, is according to each movement, the interaction of each one in the world.

In this thesis, I decided to express my own history with the goal of emphasising, influenced by enactivist theory, that for me knowledge is in a constant loop between each person and their own world. There is a chain of actions that could be traced through observation and patterns of interaction because "every act of knowing brings forth a world" (Maturana & Varela, 1992, p. 26).

3.3 Enactivist theory and cognition

In enactivist theory, which is considered to be a cognitive theory (Reid, 2014, p. 159; Proulx & Simmt, 2016, p. 100) with biological roots, cognition is the act of an organism interacting with its surroundings (Maturana & Varela, 1992). Thus, we are living, doing, and learning in a constant act of experientiality lived with the others in the world: "[K]nowledge is about situatedness; and that uniqueness of knowledge, its historicity and context" (Varela, 1999, p. 7; Depraz, Varela, & Vermersch, 2003, p. 156) is related to each one in interaction with the world.

Specifically, Varela, Thompson, and Rosch (1993) proposed the term 'enactive' in order to emphasise the following:

> Cognition is not the representation of a pregiven world by a pregiven mind but rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs (p. 9).

The history of different actions shows an organism's process of interactions, through which the organism makes sense of the world. Therefore, cognition is not seen inside of the body (such as neurons and synapses) or outside of the body (such as where external knowledge can be found). Cognition is an interplay between the body and its surroundings, highlighting, from an enactivist point of view, the centrality of each person's actions, as they simultaneously generate interactions and learn based on their surroundings.

I would like to synthesise my position as follows: My body can act because I can know, and I know because I act. Let me explain with a simple example. Imagine a person using an electric running machine which is initially turned off – his/her body is stood on the running belt. When the person selects a slow speed, the running belt is moving and at the same time the body is moving, knowing to walk. If the person changes the speed to a higher one, the body also changes its movement. The person suddenly will be noticing through his/her action on the machine, that there needs to be a change in the rate according to the movement of his/her legs. What is happening in this episode? The person can act (for example running, jogging or walking) on the electric running machine (which is part of their surroundings) because of knowing how to operate on the machine. At the same time the person knows how to operate on the machine, because of interacting with it.

In this acting in the world "cognition is, as I would claim, the *bringing forth of a world*" (Varela quoted in Gumbrecht, Maturana, & Poerksen, 2006, p. 37 emphasis in original). A world can be situated in the experientiality that each one of us has through the interaction. In the example described about the running machine, the person brings forth the world knowing how to operate the running machine through the exercise made (the type of interaction) with his/her body and the machine. In this context, what she/he learned is associated with the interaction maintained through time on the machine. Over time, doing the same exercise with his/her body, she/he has not to ‘think’ how to operate the running machine, because now he/she knows how to do it.

Finally, when an organism or person does not know something but gains knowledge at the moment of acting or doing, that is gaining knowledge through interaction as in the example illustrated on the electric running machine.

3.4 Autopoiesis

Taking into account the biological roots of enactivism, the phenomenon of autopoiesis says that an organism is capable of maintaining itself through its own network, generating itself (Maturana & Varela, 1992; 1980).

From the enactivist perspective, cognition “is seen as a property of all living systems” (Reid, 2014, p. 159) that happens due to interactions between organisms; in this study, for example, it is due to the teacher and students interacting and amongst students interacting with their surroundings.

In this sense, Maturana and Varela (1992) proposed the following that “characterizes living beings in their autopoietic organization” (p. 47). The process of self-generation of meaning appears in an innate way for everyone who is interacting (for example, in a conversation) because “the worlds that we live as human beings arise through our living in conversations” (Maturana, 1997, n.p.).

3.5 Autonomy

Varela (1981, p. 20) expresses that autonomy of a living being "means, literally, self-law". In accordance with that, "a system is autonomous if it can specify its own laws, what is proper to it" (Maturana & Varela, 1992, p. 48) and, therefore, the autopoiesis of a living being (i.e., self-reproduction) results in the consideration of that living being's autonomy (Maturana & Varela, 1992, p. 48).

Other authors, such as Thompson and Stapleton (2009, pp. 23-24) suggest a characterisation of an autonomous system (for example the immune system) that could be seen abstractly or concretely. Abstractly, the authors give three conditions for an autonomous system and concretely, they explain about "its energetic and thermodynamic requirements" (p. 24).

My intention is not to observe whether the system is autonomous. For that reason, I am not reviewing the characterisation mentioned above. I began with the premise, "the mechanism that makes a living being an autonomous system is autopoiesis" (Maturana & Varela, 1992, p. 48). A teacher, a student and myself are autonomous because each one is an autopoietic system, a human being.

The autonomy of a human being, for me, is becoming explicit through the decisions[4] that he or she makes within a set of interactions that entail options (based on Varela, 1994). The organisms "manifest themselves in particular actions and in their

[4] The word decision "is merely the fact we are seeing a multitude of possibilities which emerge at a given moment and then, finally, the organism decides to go in one direction or in another. We can simply replace the word decision with achievement of a process of choice […] we can also replace the word decision with the word selectivity" (Varela, 1994, n.p., translated from French).

appropriate environment" (Varela quoted in Gumbrecht, Maturana, & Poerksen, 2006, p. 37), showing his/her "self-law" expressed by decisions that could be noted by an observer.

The actions of each organism (e.g., teachers and students' autonomy) are observed through the decisions that the organism makes, showing that being a "researcher/observer" (Maheux & Proulx, 2015, p. 215) is a unique way to see the world and live in it.

The decisions made by any organism naturally involve interactions that could be traced by an observer looking at the surroundings where the interactions take place. For example, in Conway's *Game of life*, which is a game about cells' movements there are interactions amongst the cells according to the decisions made by the player under the rules of the game. The movement of the cells and therefore the surroundings can be traced by an observer of this game.

3.6 Conway's *Game of life*

In this section, 3.6, I present Conway's *Game of life* to illustrate the difference between surroundings and environment. This game is not being used as a metaphor for interaction between unities, nor for social systems.

The *Game of life* is a single-player computer simulation of a two-dimensional square grid with blocks that represent cells. The game starts with a group of cells and can be played by writing by hand or creating "a computer simulation of simple cellular 'life' governed by simple rules that give rise to complex behavior [sic]" (Schleicher, 2013, p. 567).

If writing by hand, the player needs to observe his or her first configuration. This refers to how the blocks representing cells are allocated in the grid, as shown in figure 4 below:

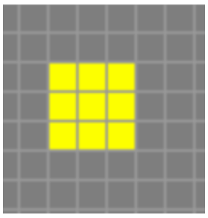


Figure 4: Example of a two-dimensional grid of a square with nine cells represented by each yellow square.

The next movement of the player is therefore to write a new configuration for the group of cells, which is done according to three possible states: survival, death and birth (which comprise the rules of the game).

Gardner (1970, p. 120), in a publication about mathematical games, highlighted the work of John Conway, the game's inventor, to be that of a remarkable mathematician. Gardner described the game's three rules as follows:

(1) Survival: Every cell with two or three neighbouring cells survives into the next generation.
(2) Death: Every cell with four or more neighbours dies (is removed) from overpopulation. Every cell with one neighbour or none dies from isolation.
(3) Birth: Every empty cell adjacent to exactly three neighbours – no more, no fewer – is a birth cell. A counter is placed on it at the next move.

The first move is the group of cells in the example given in Figure 4. The next configuration of the cells takes account of the rules of the game and requires that deaths and births occur simultaneously, as is shown in figure 5.

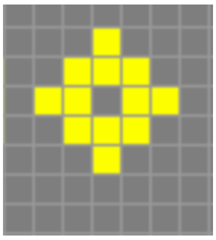


Figure 5: Two-dimensional grid of squares with eight cells, each represented by a yellow square taking account the movements survival, death and then birth.

However, if the reader wants to play the game following the order birth, survival and then death, the configuration will be as shown in figure 6:

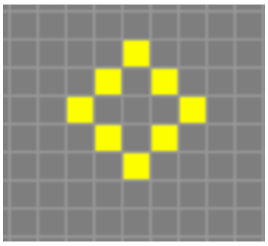


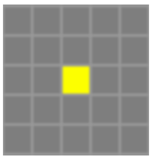
Figure 6: Two-dimensional grid of squares with eight cells, each represented by a yellow square. To draw this diagram, I have considered the order first birth, next survival and then death.

In addition, the number of configurations will depend on the first configuration that the group of cells has been written on the two-dimensional grid and of course the rules of the game and the order adopted (i.e., birth, survival and death or survival, death and birth) for each group of cells.

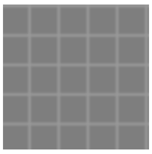
The next figures (*a* and *b*) are extracted from the example given by Gardner (1970, p. 120) in the publication about mathematical games mentioned previously, showing changes under a series of movements in order, first survival, then death and finally birth.

a

Move zero (when the configuration of the cell starts).



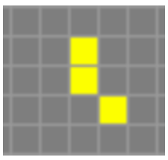
First move



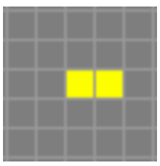
The cell died because of isolation.

b

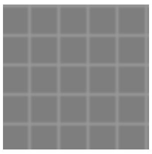
Move zero (when the configuration of the cell starts).



First move.



Second move.



The cells from the first moves died because of isolation.

Let us look at example *b*. The cell allocated on the top dies, because the cell on the top has one neighbour (the one just below it). The bottom cell dies because of isolation. The cell in the middle survives for the next generation because it has two neighbours (one on each side, top and diagonally down). In addition, a birth is happening because the empty cell adjacent has three neighbours.

The change in example *b* between the initial move zero to the sequenced moves, and the surroundings where the change took place, can be observed as follows (see figure 7).

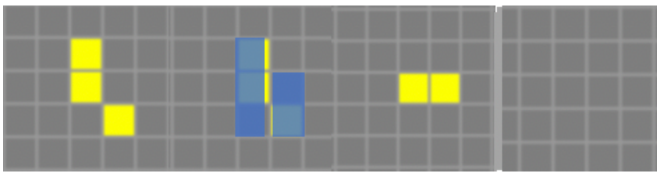


Figure 7: Two-dimensional grid of squares with group of cells each represented by a yellow square, under three movements. The blue shadow represents the surroundings from the first to the second moves of the group of cells.

For me, knowing the surroundings allows me to observe where the movement took place, or, in the case of my study, where the interactions took place among the teacher and their students, or between the students themselves. Evidently, and quite the opposite of the rule governing the possibilities of the game, in the interaction between persons, the surroundings entail a world of possibilities where the autonomy of each participant can be observed.

3.7 Surroundings

What comes from knowing the surroundings? Knowing one's surroundings allows where the interactions take place to be observed, showing how "organisms [or persons] create their own experience through their actions" (Hutchins, 2010, p. 428). Simultaneously, an organism shapes its surroundings. The surroundings are defined by the context in which actions are being performed (i.e., where the person and object, or the person and another person, are interacting).

Considering the biological roots of enactivism, presented in the work of the scientist and biologist Varela and Maturana, for me Beer (2004) illustrated physically how we can observe the surroundings. Based on the idea of the *Game of life*, he identified a glider (interactions and configuration of cells), and in doing that, also identified its environment. For example, Sapin, Chabier, Baileux and Collet (2007, p. 5) in their article demonstrating the universality of a new cellular automaton, show how, in the square (see figure 8 below), the glider 'moves' from a random configuration of cells.

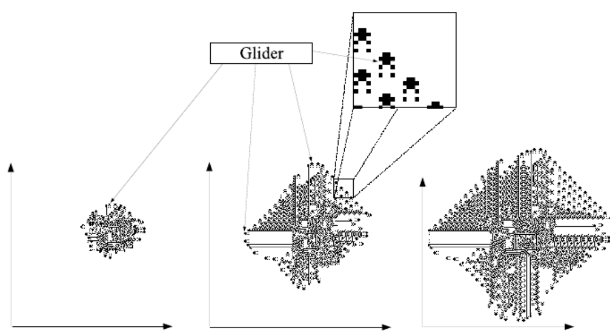


Figure 8: Glider "moves" in the square. Image from authors Sapin et al. *(2007, p. 83).*

For me, through identifying the interactions of each person, the observer can identify the surroundings (where the interactions are taking place) and, therefore, the environment, because the last one is just the set of surroundings of each person in which interaction took place (shown by the cross-hatching in figure 9).

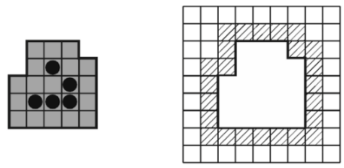


Figure 9: All interactions between the two (environment and cells, represented by black dots) takes place through the interface shown with cross-hatching (Beer, 2004, p. 315). Image also from author Beer (2014, p. 315).

The surroundings of a particular person are not the same surroundings as those of the others, because the surroundings are according to the interactions of each person. The autonomy of each person, shown in the decisions taken (see chapter 3, section 3.5, pp. 30-31), makes the surroundings unique.

In a mathematics classroom, although teacher and students are interacting in the same place, for example, the teacher can say, “Solve this equation” to their students. The students can start some mathematical actions after that instruction. The formation of the particular action of each one (students in this case), is what I see as surroundings. At the same time, for me, the environment in the classroom is the configuration of each surrounding.

In this context of surroundings, there is a constant loop between each person and their surroundings (see chapter 3, section 3.9.2, structural coupling, pp. 45-46) and this can happen because there is an interaction amongst them. Going back, to the example in a mathematics classroom described in the paragraph above, a student may disturb the surrounding of the others, through a question (a type of interaction) and this happens because the students are not isolated in that classroom and can be influenced by the others.

However, the student is the 'protagonist' of their own actions, trying to solve the problem according to his/her way to make sense of their doing.

Finally, surroundings and interactions between persons cannot be seen as divided ideas. When cognition takes place from an enactivist point of view, they work together, because 'the building up' of the surroundings is through interaction in recursive actions.

3.8 Perception and interaction

> *What we perceive* is determined by *what we do.*
> (Nöe*,* 2010, p.1, emphasis in original)

We are organisms that recurrently engage with our surroundings, including objects in multiple ways; for example, we can use a pencil to write notes during a lecture. In that action of taking notes, we are choosing a first level; we have decided whether we want to write with a pencil or with a laptop's keyboard. On interacting with the speech heard, we are also deciding which part of the lecture we want to take notes of. Thus, the process of making choices shows our autonomy to make decisions in our surroundings. At the same time, "*what we perceive* [as a teacher or students] is determined by *what we do* (or what we know how to do)" (Nöe*,* 2010, p.1, emphasis in original).

For mathematics classrooms (including teachers and students), it is not surprising that, after reading a mathematical word problem, the teacher can ask questions about the problem and receive different answers. The comments will not always be the same because in the interaction with the surroundings, our perceptions guide our action (Maturana & Varela, 1992, p. 173, see the example of the running machine described in section 3.3, *Enactivist theory and cognition* in the current chapter, p. 28). What is perceived in the moment of running on the machine, is considered to be a specification of what each one is doing, (based on Maturana *et al*., 1980, p. xv), for example, the person can note slow or high speed (a type of specification) on the running machine.

In addition, what is perceived through the interaction, allows the observer to distinguish and categorise. For example, if I am driving a car to my job on a particular route and I hear on the radio there is traffic congestion on route A, which is the route that I usually take going to work, I start to categorise alternative routes and the time taken between these routes, also considering the option to follow my usual route. Finally, I make a decision (that shows my autonomy in how I perceive the world in that moment) to stay on the route or perhaps change to another alternative, illustrating unique ways of acting. The decisions that we make are when we are actively operating in determined surroundings and "participating in the generation of meaning in what matters to us" (De Jaegher & Di Paolo, 2007, p. 488).

Another example would be the story from the section *My journey* (see chapter 1, section 1.1, p. 7) about working with undergraduate students in the technical institution. A change from a mathematical problem to another occurred. It was not planned ahead; it happened naturally—an operation determined by our surroundings and an action of acceptance about what we were doing mathematically. I am not saying that we did not know that we would be working in algebra and with problems in a textbook; nonetheless, the mathematics that we made, literally, came in that moment.

I must point out that this act of doing mathematics means acting according to our personal history, our autonomy and the perceptions we are generating in the interactions. However, we can reach a mutual agreement (an action of acceptance) according to the experience lived in our surroundings and environment.

When interacting with surroundings, a person's autonomy can be seen by an observer through what that person is doing, the decision (see chapter 3, section 3.5, pp. 30-31). This interaction provokes what we usually call the 'environment', which is no more than a set of interactions of each organism that at the same time creates their surroundings.

So, how could our interactions with our surroundings be characterised? To answer that question, Maturana and Varela (1980, 1992) direct our attention to the relationship between the structure of an organism and its surroundings.

3.9 Organisation and structure

Maturana and Varela (1980, 1992) pointed out how the structures of a particular set of objects in the same class can be determined by its component parts. For example, I am observing an object which has three components such as: legs (support), surface (seat) and a back, in which the back is connected with a surface, and the surface is held by four legs, each of these legs is situated in parallel.

The structure of this particular class that I am observing, a chair, is determined by the components that I am observing, including the relations that I can establish with them, that is to say, flex my legs and knee, support my foot on the floor, press my bottom on the surface and I hold my back to finally sit on that object. Performing those distinct actions in relation to the components (i.e., back, legs, support), I can note the class organisation. To Rudrauf, Lutz, Cosmelli, Lachaux and Le Van Quyen (2003), the organisation is the identity of a particular object or the self.

How can I name a classroom as a mathematics classroom and not something else in a school? I observe the manner of how the components behave, the students and the teacher (their identity according to Rudrauf *et al*. (2003)), making me decide on the name mathematics classroom and not something else. But are all mathematics classrooms the same in the school? The answer to this question, naturally, is, "Not". Depending upon how the components (teacher, students, seat, desk, whiteboard, persons and objects) are interacting, although the components can change, the organisation will still be the same (based on Maturana & Varela, 1992, p. 47). The mathematics classroom can still be recognised as a mathematics classroom and not something else, let us say, a biology classroom.

This identity is not only determined by the organisation, because organisation and structure work simultaneously. In this sense, components and their relations in the organisation determine the structure "that makes its organization real" (Maturana & Varela, 1992, p. 47). In the example before, even though chairs can look different, it was a chair. Here, we see a mathematics classroom and not something else. Therefore, organisation and structure do not work separately; they are dependent on each other based on their components, even considering that the components are changeable (i.e., in the object, chair, for instance, made of plastic, wood or metal) but maintain a 'certain order' that allows us to observe the organisation (based on Rudrauf *et al.,* 2003).

Knowing the relations amongst the components, I am distinguishing between the 'relation' and the 'components'. If I know the component and their organisation, I can observe the object's structure; if I do not know the components, I can specify neither its relations nor its structure because I cannot distinguish what they are.

In particular, the way that each person behaves, their autonomy expressed by the decisions that each one makes, shows their identity, the class of organisation that belongs to each one.

Let me explain further with the following example of an organisation and structure.

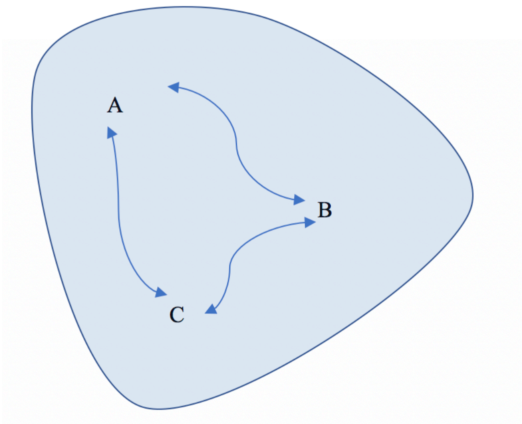


Figure 10: A, B, and C (in no particular order) represent persons speaking in a group. The blue arrows show the interactions amongst the participants in the group.

In a learning situation, if I am walking on the street and I observe a group of people at a street corner, I might see that one of them is speaking in front of the group.

The one who is speaking might gesture with his or her finger, (i.e., pointing out a statue) and the others might be performing other actions (expressed by the arrows in figure 10), such as, asking questions, raising their hands, and observing a piece of paper.

From my observer perspective, I would say that they are engaged in a lesson because I noted some coherent actions from the components that emerged amongst the members. Based on the changes in their actions, I was able to distinguish a lesson—a specific class of behaviour amongst the participants of a group, a person speaking to others, the others replying and actively participating in the generation of meaning for each one, taking notes, making some gesture with their face, for example moving their head in a sign of approval or disapproval.

Therefore, I am not referring to the physical space in which this organisation happens due to their relationships; people's actions can occur, for example, in a park, classroom, museum, home, or street.

However, I must recognise that my interpretation of the common actions being performed on the street corner is influenced by my experience of being a teacher. The way that they can generate their relationship is dependent on them. For that reason, not all learning situations are equal; the place, the students, the teacher, and the kind of conversations can change. In other words, the dynamics of their relations and their structure can change (expressed by blue in figure 10), but their organisation cannot change because it is still a learning situation (due to the existing relations between them).

3.9.1 Action amongst organisms

From an enactivist point of view, "each organism creates their own experience through their actions" (Hutchins, 2010, p. 428) that they make in their surroundings that are constantly generated in interaction with the world.

In this context, their actions "are determined by internal dynamics, not external pressures" (Drodge & Reid, 2000, p. 252) because each organism is autopoietic and therefore autonomous. As a consequence, although we can receive perturbations from the world,

> interaction is not instructive, for it does not determine what its effects are going to be (Maturana & Varela, 1992, p. 96)

For example, even if a teacher has taught equations to their students in some way, it does not mean that the students will be solving a particular equation in the same way that the teacher did, because of the type of decisions each student made, in his/her own self-organisation and their interactions with their environment. In this context, "the children do not learn mathematics in school; they learn how to live together with a mathematics teacher" (Gumbrecht, Maturana, & Poerksen, 2006, p. 26).

3.9.2 Structural coupling

> We speak of structural coupling whenever there is a history of recurrent interactions leading to the structural congruence between two (or more) systems. (Maturana & Varela, 1992, p. 75)

Through interaction, each human being is generating his/her surroundings and these surroundings are also composed by other organism at the same time. As a consequence, all of them are interacting in the same environment.

These interactions between them involve a trajectory of perturbations and a history of experiences that could trigger each organism to adapt to their surroundings (based on Towers & Martin, 2015, p. 255). In line with the historicity of each person, this means that particularity of knowledge in a determined time is through the actions that made it. Towers and Martin (2015, p. 254) noted that in these pathways of perturbations there

> is a breath of possibilities open to each participant at each moment and the pathways taken are dependent on what has gone before, what each word, gesture, drawn line, facial expression, etc suggest to other hearers/viewers and what moment-by-moment response is made as a further action.

For example, within the context of a mathematics lesson, through the interactions, including with other classmates, teachers and tasks among other things, we are in a constant process of adapting to the situation ahead and receiving perturbations from, for instance, tasks, words and sounds. These influences can have triggered changes in the students and teachers. However, it is important to note that these changes (an effect) can be triggered by other organisms or vice versa, because "structural coupling is always mutual" (Maturana & Varela, 1992, p. 102). The environment and organism show change only if their structure allows it, because our changes are determined by our structure (who we are) and are not determined by the environment (see chapter, 3, section 3.9.3, structural determinism, pp. 46-47).

3.9.3 Structural determinism

> [T]he changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system* (Maturana and Varela, 1992, p. 96; emphasis in original)

As previously mentioned, the structural coupling between organisms and their surroundings creates a world of possibilities that triggers changes in each organism; however, the structure that receives the perturbation determines the changes or the shift because each organism is autopoietic and therefore autonomous.

To make sense of disturbing agents and the disturbed system in a mathematics classroom, one possibility is observing the behavioural changes of the person who is subject to a perturbation, such as responding to a question asked during a conversation. Behavioural changes can occur in each conversation, but they are triggered by that which is received (i.e., the person who received the perturbations). This might explain how the same question can sometimes trigger different answers, as “emotion is not determined by the situation in which one finds oneself but by one’s structure” (Drodge & Reid, 2000, p. 252). Evidently, there is the necessity for some interaction within the environment to trigger or provoke this behavioural change through a received perturbation and this behavioural change can be noted by an observer.

There is a remarkable example mentioned by Maturana & Varela (1992, p. 99). They said, "it is obvious that a compact car crashing into a tree may undergo a destructive interaction, but this would be a mere perturbation for a tank". Based on that idea, if a car is crushed by a tree, both will be changed, but the kind of change is different, and this is because it depends on the structure of each participant in the interaction. The tree might lose some leaves and the car can break lights, but both change, the actions of change depending on each structure (car-tree).

This idea of structure and change can be linked with a study conducted by Guberman, Barabash, and Mandler (2016, p. 4) on training elementary school teachers in mathematical modelling; which evidenced that two teachers, under the same conditions, conduct the lesson in different ways, as one tended to be more open with students, while the other was less interactive when they teacher supported their students.

3.9.4 Coherent behaviour

> The life of a living system appears to us observers as coherent with its circumstances of living, even when we see what we did not expect. (Maturana, 2000, p. 460)

When an interaction takes place between persons, in this study mathematics conversations between teacher and students and between students themselves, different actions happen. This is not surprising, because, as I discussed before and based on my enactivist approach, each operation in the world, through a question, a touch, a stimulus in general from the environment, triggers a different response in each one. This response is where we and our senses are actively living in that particular moment.

The environment and the surroundings in which each person is interacting, (i.e., a student solving a mathematical problem in the mathematics classroom) is important because the observations allow the observer to recognise what is the dynamic behaviour in that moment, how "a living system appears to us observers as coherent with its circumstances of living, even when we see what we did not expect" (Maturana, 2000, p. 460).

Considering that "learning arises from learners" (Proulx, 2009, p. 270) through a coherent action that allows the learner (teacher or student) to continue acting in their environment, I (as an observer) can note how this learning starts to emerge. Clearly, these coherent actions can be observed in different ways, for example, in relation to a physical object or through maintaining a coherent action within a conversation.

Pirie and Kieren (1989), based on Maturana and Varela (1987, re-edited 1992), define effective action as,

> if one wishes to see if a person knows how to play the piano, the person might actually be asked to play. Then the observer determines if this action is effective (p. 8)

For me, the way in which action is seen as effective by Pirie and Kieren goes is the same idea as coherent behaviour. The observer can see how the person is doing the action and then the observer determines if this action is coherent or is in tune with the circumstances for him/her, like the person who is playing the piano in the example given above.

Lozano (2004, p. 128) quotes Maturana (1987, p. 66) as saying,

> [I]f my problem is cognition itself, or knowledge, and I recognize [sic] knowledge by seeing adequate conduct, then my problem will be to identify adequate conduct or to show how adequate conduct arises.

She describes effective behaviour as those "actions that were adequate in each one of the classrooms" (p. 128) and reports that her method is "describing conduct which I found to be effective when I observed the lesson, and also by analysing the students' responses to interviews as they commented on what they did in the classroom" (p. 128) Therefore, following Lozano, an effective behaviour happens when the observer notes "adequate actions" in the persons (for her the students) who are interacting in a way that allows continuing interactions. To her (2004, p. 129) an adequate action can be observed, for example,

> [I]n a given classroom, I found that asking questions was effective for the students, giving rise to dialogue and further interactions. In another classroom, a question asked by student was considered to be an interruption and therefore was not encouraged by the participants. Behaviour that is effective in a certain environment will normally be promoted, and hence occurs frequently.

It is possible to observe adequate actions, for me, when the interaction of the persons (i.e., students) allows each one maintaining their action, in the example above given by Lozano, questions that allow further dialogue between the students.

From my point of view, Lozano (2004) with "the adequate action of" as well as Pirie and Kieren (1989) with "action of" highlight the role of the observer in seeing behaviour to be adequate. In other words, the person is maintaining a coherent behaviour that could be observed by the observer.

Instead of using the words effective action or effective behaviour, mentioned by the authors above, I would like to continue to use the words coherent behaviour in order to be in line with the definition given by Maturana (2000) in the beginning of this section (p. 46).

Let me explain what coherent behaviour is to me with the next example. A teacher wrote this procedure on the whiteboard x + b - b = c - b, x = c - b to solve the equation x + b = c. A student, after spending time solving the equation with the teacher's procedure, suddenly started to write just x = c – b, which was not the teacher's procedure. For example, for the equation $x + 7 = 12$, he/she just writes $x = 5$. What happened in this interaction between these students and the prompt, "Solve the equation"? There is a coherent action connected with what this particular student is doing (or living) and also connecting with what the teacher and the other students are doing: solving an equation. However, the coherent action in solving the equation does not mean doing the same as the others, because each one is unique, and that 'uniqueness' is shown in their autonomy and the way each one, or in this case, this particular student, finds a way to act on it. The student has a way to solve this type of equation that works for her/him. Perhaps, the next time, this student will not 'think' about how to solve an equation because he/she has learned a way to do it.

Until now, I have mentioned how a coherent action can be observed in line with the environment or surroundings, however, what is an incoherent action? Or when is it possible to observe an incoherent action? For me, the answer to these questions resides in the relation to the environment and the context lived, that is to say, when there is a break between actions and their environment.

I am not speaking about break and change as synonymous words. What I am saying is that a break shows a gap to an observer between what has been done and what follows. To observe changes in behaviour, I need to look at pathways, or sequences of behaviour over time, to see that before and after are still connected.

3.10 Enactivism and mathematics education

> The cognitive science of mathematics asks questions that mathematics does not, and cannot, ask about itself. (Lakoff and Nuñez, 2000, p. 7)

In this sense, as the theory of learning through action, enactivism "emphasizes the dynamic interdependence of individual and environment" (Towers and Martin, 2015, p. 249) to bring forth, in the eyes of the observers (in this study myself), knowing and mathematical knowing through the body, mind and the constant interactions with their unique surroundings.

Enactivism in mathematics cognition has been around in different ways for over three decades. A general description of its influence and branches is not possible here due to space limitations and the purpose of this document. However, I would like to highlight that, from my point of view, the consideration of the body and its interactions with the environment have influenced mathematics education, through no separation between inner and outer (the environment). For me, the study by Ärlebäck and Albarracín (2017) is an example of focus primarily on mathematics itself. They used a mathematical modelling perspective to develop a classification of Fermi problems, which are problems related to mathematics estimation i.e., “How many drops of water are in Lake Erie?”[5] In this case, the focus is on the mathematical problem itself, the definitions and description given, that is to say, focussed on mathematics in isolation.

However, an illustration of the non- separation between inner and outer (the environment), for example, is the view that “context is not merely a place which *contains* the student: the student literally is part of the context” (Reid & Mgombelo, 2015, p. 177, emphasis in original) and the teacher as well. Therefore, what each student or teacher is doing with their mind and body in a mathematics lesson is also the context (or the environment), which is the surroundings generated in each interaction.

An illustration of the students and their context can be observed in work about attitude toward mathematics from Hannula (2002). He analysed the case of a secondary student Rita and how her emotions related to mathematics changed from a negative to a positive attitude. Another illustration is the longitudinal study, with students from years 6, 7 and 8 in two schools, where Lozano (2008) reported on a characterisation of algebraic learning. She found that if a student has a need for the use of algebra, acting algebraically became part of their behaviour showing procedures,

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[5] Online at http://www.physics.uwo.ca/science_olympics/events/puzzles/fermi_questions.html, accessed on 20th June, 2019.

explanations and justification within his/her own mathematical structure (Lozano, 2008, p. 329).

Lakoff *et al.* (2000), in the field of embodied cognition, emphasised bodily experience with mathematics concepts through metaphors. A recent study by Abrahamson, Flood, Miele, and Ting Siu (2018, p. 299), with blind and visually impaired mathematics students, suggests the use of enactivist theory as part of the design of accessible materials. The authors propose how the interactions, and therefore mathematics learning based on an enactivist approach, are in the relationship established by each student through the use of touch (and other senses) with a tactile device working with another student. They move their hands in a vertical way, receiving input "of success" and, later, they are asked to move their hands together in order to the find the place without "noise" (see figure 11).

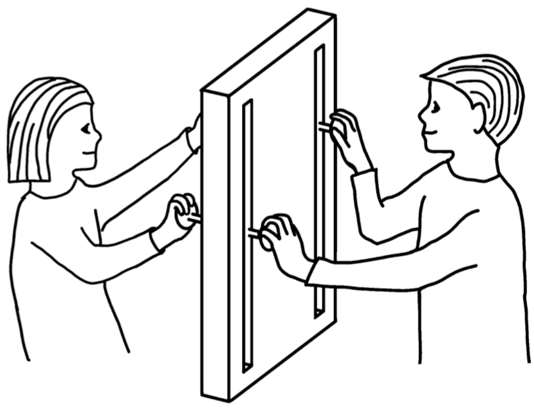

Figure 11: Students working together with a tactile device, a mathematics imagery trainer for joint problem solving (Abrahamson et al. 2018, p. 299).

Finally, as Maheux and Proulx (2015, p. 212) noted, work in mathematics education from an enactivist point of view is motivated by "how students actually *do mathematics*, replacing questions of knowledge with concern for mathematical doing alone" (emphasis in original). For me, the ideas of Maheux and Proulx can be extended to how the teacher actually does mathematics with their students as well.

3.11 Metaobservation

I came to this chapter trying to make explicit the importance of the surroundings in the doing of each person. I began to note that simultaneity of doing and knowing is firmly arrayed in our autonomy and is structurally determined, but also triggered by what we are living.

In my thoughts, I believe we are preparing to know and act in different scenarios, which always entail possibilities that can be triggered by "something else" (other actions).

The phrase "avoid judgment" started to resonate in my thinking. This began by being open to possible actions of participants in my interactions with them. I did not have expectations from an *a priori* analysis of the mathematical situation leading me to seeing, let us say, 'good' or 'bad' mathematics in their doing.

So, instead of thinking about "a supposed to know" or looking for a representation of something from before, such as previous knowledge, I am thinking, "learning to learn?" (borrowing some words from Varela & Flores, 1999) or, in particular, "How does the other know what they know?"

CHAPTER 4: Pathways of doing mathematics

4.0 Introduction

In the previous chapters, I have explained how I arrived at an enactivist position and how my way to see learning in mathematics shifted from constructing mathematics knowledge to doing mathematics.

In this chapter, based on my enactivist approach, I explain what the mathematics of doing is to me. My research question is developed: *How is the emergence of mathematical knowing shown in the details of the shifts in interactions through conversations between students and amongst the teacher and the students in a mathematics classroom*?

I conclude this chapter with a metaobservation about experience and doing mathematics.

4.1 Mathematics of doing

Based on my experience of doing and learning mathematics described previously, in chapter 1 and 2, my reflections are connected to living mathematics. In this sense, mathematical concepts or any mathematics can be known only by doing.

In the words of Romdenh-Romluc:

> "[T]he geometer does not construct their diagram in accordance with a representation of it that they have formed in advance. Instead, they see

> what to do as they construct their diagram" (Romdenh-Romluc based on Merleau-Ponty, 2011, p. 208).

Therefore, knowledge in mathematics is not about static knowledge or preconceived knowledge; it is about doing because "every act of knowing brings forth a world" (Maturana & Varela, 1992, p. 26), a mathematics world from each one, from each person's act of knowing. As a result, there is no distinction between knowledge and external reality because "knowing is doing" (p. 26). As a consequence, every act that we are doing brings to us a set of possibilities in which we are constantly involved in the options (or decisions) taken when we act. Our knowledge is not static, in a similar way, as soon as we are doing, we are knowing. But what defines what is mathematics of knowing? Or more precisely what is doing mathematics?

For me, mathematics of doing is the 'unique mathematics' which arises through the inter-action of each individual with their surroundings, which can involve a mathematics task, technological objects, classmates and teachers.

Notably, this mathematics depends on each person and his/her way to act, allowing him/her to maintain coherent behaviour as in the example of solving an equation (see chapter 3, section 3.9.4, p. 50) or moving on to doing another idea. This is like a metaphor of a navigator. When going to sea, the navigator can plan the trip, for instance, according to the weather conditions, which can change in the middle of the journey, waves sometimes calm or strong. What happens then is that the navigator changes behaviour according to the types of waves that are present in that moment. In a similar way, mathematical actions depend on what the environment brings to me and what I am bringing to the environment.

In this context, the navigator is navigating in circumstances of uncertainty, however, “we tend to live in a world of certainty, of undoubted, rock-ribbed perceptions: our convictions prove that things are the way we see them and there is no alternative to what we hold as true” (Maturana & Varela, 1992, p. 18). For that reason, we can fall into habits of certainty; for example, mathematics “is often still seen as a purely deductive activity in which perfectly rigorous formal proofs are used to produce theorem after theorem” (United Nations Educational, Scientific and Cultural Organization [UNESCO], 2012, p. 10). Occasionally, we lose the capacity for surprise in mathematics because we think that there is a ‘rule’ without meaning. How many times have we heard the rule, two minuses make a plus, said when we are multiplying two negative numbers? As a consequence, we can prevent ourselves from reflecting on how we are doing what we are doing. Then, mathematics knowledge seems to be something difficult and not dynamic, despite the fact that “all doing is knowing, and all knowing is doing” (Maturana & Varela, 1992, p. 26).

The significance of surroundings in the classroom and the experiences lived in those surroundings is part of my doctoral study. In this sense, an individual’s personal experience is an important part of my doctoral research—as well as their unique experience lived interacting with others.

In a particular study that I made regarding beliefs about mathematical modelling (Ramirez, 2017, p. 974), the literature reviewed, of teacher beliefs on learning and teaching mathematics, seemed not to have included relations with past experiences of the teacher (i.e., Stipek, Givvin, Salmon, & MacGyvers, 2001, p. 213; Handal, 2003, p. 47); again, in my thoughts, the experience of each person in their environment is calling my attention.

Some research indicates that teachers do not know how to implement the modelling process (i.e., Tekin Dede & Bukova Güzel, 2016, p. 1). So, what is happening to the link between mathematics teachers and their environment? How effective are the current training courses for mathematical modelling? At this point, each teacher's approach to mathematical modelling from their own experience has not been reported yet, nor have the pathways within mathematics in any implementation of mathematical modelling. What would taking account of the pathway of each person in a mathematics setting be like?

4.2 My research question

I began noting others shaping their surroundings in their experience of doing mathematics. The importance of the person's experience and approach was in my thoughts again, not only as a mathematics teacher but also as a researcher.

I started to consider how mathematical modelling approaches view the environment, which was one of my primary aforementioned concerns, and how the experience of the teacher was being taken into account when a mathematical modelling task was carried out? How was I taking account of living mathematics and the experiences with it, including the uniqueness of each person's pathway in this process?

Considering my attention to mathematical modelling and also what happens in particular in the new Chilean curriculum around this theme (described in chapter 8, section 8.2, pp. 118-120), I realised that my research was more than how mathematical modelling works, the pathways students and teachers take in a lesson and, perhaps, how these pathways can be linked with mathematical modelling as a cycle (Blum & Borromeo, 2009; Burkhardt, 2018).

I was concerned about observing the experiences that students and teacher have in a particular moment and the ways in which each student and teacher can approach mathematical activity. The word pathway and specifically pathway in mathematics learning for each one, started to resonate strongly in my ideas.

My attention was caught by the unique ways of doing mathematics of the teacher and students, bringing new thoughts regarding how mathematics learning emerges. Suddenly, my research was not only about mathematical modelling, it was about being in the experience of learning mathematics. I was learning about my own learning about students' and teacher learning in mathematics.

For me, the experience is firmly linked to the interactions between the participants, in this study, the teacher, the students, and the researcher in the role as an observer (based on Maturana & Varela, 1992).

However, what comes from an interaction in a mathematics classroom? What interactions take place between the teacher and the students and between the students themselves? These two questions arose after I looked back over my history as a mathematics teacher (see chapter 1, section 1.1, pp. 2-8: *My journey*), and, also arose later, when as an observer, I was researching the interactions between the teacher and students in a mathematics classroom.

While many studies (e.g., Chapman, 2004) focus on the interactions of the students and teacher in mathematics classrooms, there is little focus on how mathematics emerges from the details of their interactions, for example, within a conversation. I want to make an account based on “interaction is not instructive, for it does not determine what its effects are going to be” (Maturana & Varela, 1992, p. 96).

After this process of learning and doing my research, and taking account of a possible characterisation of the mathematics learning process through my own observations, I decided to focus on how my own learning began to evolve through the observation of mathematics knowing emerging from the details of the interactions in a mathematics classroom through conversations both between the teacher and students and amongst the students themselves, which led me to a question:

Between students and amongst the teacher and the students in a mathematics classroom:

From the observer perspective, how is the emergence of mathematical knowing shown in the details of the shifts in interactions through conversations?

4.3 Metaobservation

I began to note, based on my enactivist position, that the living mathematics, which I have mentioned in this chapter, is not more than the experience of doing it. For me, experience is doing. How can I have the experience of something without doing? Or, more specifically, how can I have the experience of mathematics without doing?

Obviously, when a person, (in this study a teacher and each student) is doing mathematics there is a unique interaction (i.e., touching, speaking, reading, using the senses in the world) with the surroundings, a particular way to shape his/her own understanding through a set of possibilities (see chapter 3, section 3.9.3, pp. 46-47) that could be observed.

The doing of mathematics is unique for each person, considering the interactions between the teacher and the students and also between the students themselves. Although the unique interaction belongs to each person, this does not mean they are acting in isolation. On the contrary, each interaction is necessary with the others (including objects) for the emergence of cognition, and therefore learning.

I want to explore the uniqueness of emergence of mathematics learning through the conversations (a type of interaction) that happen in a mathematics classroom, learning about my own learning about what they (teacher and students) are doing mathematically.

So, how do I conceive learning in the interaction? In the next chapter, I will reflect on and discuss learning by interaction.

PART TWO: Mathematics learning

CHAPTER 5: Interaction and learning

5.0 Introduction

Until now the previous chapters have shown my experiences of being involved in mathematics education, describing how I arrived at my enactivist epistemological position.

Chapter 1 is an account of how I started to think about mathematics learning. Chapter 2 presented the way in which I began to have a sense of doing in mathematics, which I derived via my enactivist position, described in chapter 3. The goal was of understanding the mathematics of doing regarding each person and their surroundings when interactions and/or actions take place.

Part One: A story of experience finishes with pathways of doing mathematics including, in chapter 4, my research question.

In this first chapter of *Part Two: Mathematics learning*, I consider interaction and learning. The actions of each person, and the interactions that emerge through action, allow organisms to act and co-exist with actions. In other words, being part of the world can trigger actions that reflect each person's autonomy (i.e., in each decision made). This chapter will discuss learning within interactions by considering historicity from an enactivist position. Distinctions within actions will be derived from myself as a "researcher/observer" (Maheux & Proulx, 2015, p. 215).

At the end of this chapter, the metaobservation, which is a re-observation of what I have done, notes that the history of interactions allows me as observer to see in more detail the actions of the persons.

5.1 Interactions amongst persons learning

> The interactions (as long as they are recurrent) between unity and environment will consist of reciprocal perturbations. In these interactions, the structure of the environment only *triggers* structural changes in the autopoietic unities (it does not specify or direct them) […] The results will be a history of mutual congruent structural changes. (Maturana and Varela, 1992, p. 75, emphasis in original)

In the quotation above, Maturana and Varela stress the relationship that happens between each person and the environment, through a reciprocal interchange of actions amongst persons (named as unity by the authors). From my point of view, this reciprocal interchange highlights the unique structural changes experienced by each person at the moment of interaction.

But, why is the autopoietic unity or person experiencing a change in the environment? Paraphrasing what Maturana and Varela said in the quotation above, each time the person is interacting he/she is receiving perturbations. The learning in this interaction is through the interchange of perturbations (i.e., the teacher asks a question) that makes up the chain of actions in which the person (for this study teacher, students and me as an observer) shows what makes sense for her/him in his/her action. Naturally, each student's answer to the teacher's question or perturbation received is unique because it is according to the decisions taken by the person and his/her own way of living in the world.

From an enactivist perspective, the learning of the person happens in interaction with his/her own surroundings, "knower and known, both organism and environment, co-evolve in a constant process of becoming" (Proulx & Simmt, 2013, p. 66). This is one of the observable changes emerging from the interactions. At the same time, this involves the surroundings of others (see chapter 3, section 3.9.2, p. 45, *Structural coupling*) which could trigger actions according to the structure of each person.

5.1.1 Interaction in the mathematics classroom

In the mathematics classroom, each action of the teacher or student is not isolated but instead relates to their actions in their surroundings, which flourish when the interaction takes place.

This includes the interactions or changes of each teacher or student through a coherent process of adapting to what they are living in their environment. For example, the teacher goes to the class, presents a mathematical problem and the students reply to this problem through actions such as asking questions, writing, silence and gesturing with their bodies. The teacher replies with other questions, perhaps, or other types of movement such as going to the whiteboard and writing. The chain of actions amongst the students and teacher begin to flourish. However, although the interactions could trigger change in the teacher and each of the students (as the question does in the example above), each of them (teacher or student) directs the change through their autonomy (associated to our structural determinism), acting differently. This is what I want to observe.

In the classroom context, and considering my enactivist position, it seems to be necessary to consider details observed in the interactions amongst persons, especially in the classroom. As an observer, I will be noting and specifying the observed changes emerging through the interactions. For instance, I can observe my baby holding his foot with his hands, stretching and flexing his legs. After a while, I can say he is moving his legs and foot, specifying the type of activity he is doing.

Conversations between students and a teacher are recognised as a usual way for them to interact. Through conversations (an action generated by persons who are interacting, acting being not one-way or linear) can begin a process of structural changes that allow me, as observer, to make sense of their actions. For example, an observer is seeing two persons speaking. Within the interactions of those persons, the observer can note the structural change provoked by the questions or answers made. Perhaps a kind of question or answer in the conversation could be evidenced by the observer to say the focus in this conversation was happiness and through their actions specify it as a friendly conversation. Of course, I am aware that in other cultures (not the Chilean culture) happiness is, perhaps, expressed with other actions. In a study about the conceptualisation of happiness among Emirati and other Arab students they see the happiness:

> as a collective state generated through relationships with family and social groups, rather than through the self. It was also defined as an emotion and approach to life and involved religion and goal setting (D'raven & Pasha-Zaidi, 2015, p. 1)

Although the definition given in the quotation above about happiness does not explicitly show actions of the participants that could be seen as happiness, to me there are actions, for example, in the relationships with the family and social groups that an observer could see as happiness.

5.2 Learning via interaction from an enactivist perspective

Lakoff *et al.* (2000) noted that "[m]athematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment and our long social and cultural history" (p. 9). In this regard, the only mathematics that we can know is the mathematics that we are doing (that which comes from ourselves) and the sense that we are making is the things that we are doing in our living. What we are doing is what we are living. The way in which we can learn and make sense of things is in accordance with our structure (structural determinism) and the structural changes (different behaviours) that we can experiment with via constant interactions with the environment when we live the process of being there, i.e., acting, doing mathematics.

In this process, "the organism as a whole is its continually changing structure which determines its own actions on itself and its world" (Reid *et al.,* 2015, p. 173) showing their autonomy and the way to act in the world. For that reason, there is not something planned ahead; rather, the organism acts and reacts to the events experienced: "Learning is modification in behavior [*sic*] due to experience" (Davis, 2004, p. 181) that happens through the interactions.

In a mathematics classroom, learning is happening via the interactions of each organism with the surroundings (based on Proulx & Simmt, 2016), which simultaneously involves the surroundings of others (structural coupling) and triggers actions. In this place, "one thought sparks another, and an idea spreads through the room; knowledge in this setting seems to exist in and consist of the participants' patterns of interaction" (Davis, 1995, p. 4). For example, a student can ask a question, and either the teacher or other students can reply. Other students can then become involved in the conversation that stems from the original question and so on, causing learning in this setting to emerge from the interactions of the participants involved in the conversation. This shows a chain of actions and their historicity (see chapter 3, section 3, p. 27; and chapter 10, section 10.3.1, pp. 200-201) when students are doing mathematics. However, in the same context, if there is another person who is not participating 'actively' (i.e., not taking part in the conversation), it does not mean that he/she is not involved in that environment. Every action, including silence, observing and listening, involves being a part of the world (see chapter 3, section 3.9.2, p. 45) and extracting (or bringing forth) what makes sense to us. An observer who is tracing changes in behaviour can note autonomy through the decisions that the person is doing.

Enactivist theory clearly establishes how knowledge is triggered through the history of our interactions due to structural coupling, which allows us to observe how learning occurs during interactions. This contrasts with "conventional explanations [that] view learning as process by which a learner internalizes knowledge, whether "discovered," "transmitted" from others" (Lave & Wenger, 1991, p. 47), in which there is an established division between the world to be discovered and what we are living.

In addition, as Reid (2014) explained, the enactivist position "has particular strength in describing interactions between cognitive systems, including human beings, human conversations and larger human social systems" (p. 137). This description enables researchers to observe and point out situations, such as what happens when teachers and students perform mathematics in their own settings, such as a classroom.

In this setting, such as a classroom, as researchers, it is important to say, we are also participating when in the role of observer, because we are also interacting with our surroundings and the environment when the interactions are taking place. As Maturana noted: “It is, after all, an observer who observes the observing” (Maturana & Poerksen, 2004, p. 37) that is not limited just to the moment of being there within a classroom, but also happens again when we are observing what we have observed.

How can interaction be observed from this position? From my perspective, it is similar to the illustration by Maturana and Varela (1992) in figure 12. This is referred to as structural coupling, even though the recurrent interactions may drift in two directions (Maturana & Varela, 1992, p. 88).

The left side of figure 12 (blue shadow) illustrates the concept of structural coupling between cells and the recurrence relation; the right side shows how these recurrences between cells could arrive in two different directions of (a) symbiosis and how (b) a recurrence of coupling in each cell maintains their unique border while simultaneously establishing a new special coherence because of the coupling.

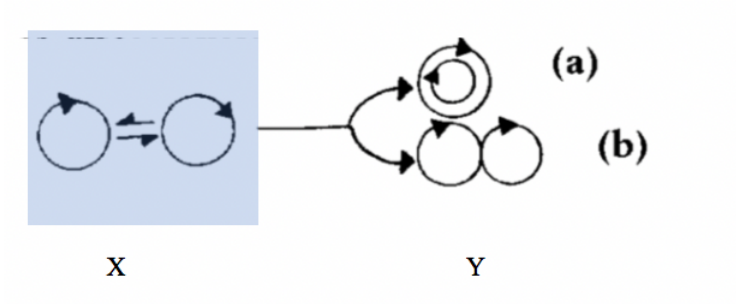

Figure 12: Structural coupling through recurrent interactions may drift in two directions (a) and (b) (Maturana & Varela, 1992, p. 88). Image from authors Maturana et al. *(1992, p. 88, blue shadow added).*

In this context, I would like to create a metaphor regarding the behaviour of the cells and the behaviour between persons that may happen when the interaction occurs.

Let us suppose that person X is interacting with person Y as shown in figure 12. They have a recurrent relationship through a conversation, a type of action that arises from the world in which we live. For example, when the conversation begins, with the person who initiates the dialogue speaking about fractions, say, one person is receiving a prompt, such as either a question, word or expression, related to the specific theme addressed (fractions) to which the other can reply. This interaction could lead in two directions: (a) both persons appear to be 'speaking the same' or following a consensual agreement in their behaviour, which could be viewed as both being in 'harmony', seeing, for example, fractions as being part of whole or (b) they have each established a unique way of understanding in their conversation. This is not 'speaking the same', they are not referring to fractions as being part of a whole directly, however, each can address the theme in alternative ways, such as, fractions as percentage or fractions as proportion. In this context, these are still coherent behaviours, under the same theme, fractions, but with alternative pathways.

In both cases of recurrent behaviour, persons X and Y are structurally coupled all the time (because we are part of the world and the world is part of us). From the prompt received, there would be changes in them, that are certainly not the same in each one; yet, each person is generating their own transformation through this interaction (the conversation, their hearing, their speaking), and their decisions, i.e., in the example above a person is speaking about fractions and percentage and the other speaking about fractions as proportion, reflecting their autonomy, which is the unique way that they are operating in their surroundings. How can these interactions be 'illustrated' in the actions performed by the observer?

One possibility is noting a change in behaviour, because, as I mentioned previously and based on Davis (2004), learning is understood to be a modification in the behaviour of each person. Similarly, Coles, Liljedahl, and Brown (2017) have noted that, "learning is indicated by a shift in these patterns of interaction, [amongst persons] by seeing differently and therefore making new distinctions in a particular context" (p. 258).

When the interaction happens, it allows the observer to establish a border, a distinction. In the conversation, that border is linked to modification or change in the conversation between the participants.

5.3 Learning as coherent behaviour

From an enactivist perspective, learning emerges through the interactions between a person and the environment, including interactions with objects, persons, and animals as I discussed previously in the section above.

When the interaction is taking place, this can be labelled as effective learning, if the person is living a structural change, doing something differently, which allows them still to be acting in the environment (based on Maturana, 1987). I will name this coherent learning in order to follow the same idea (mentioned in chapter 3, section 3.9.4, p. 47) about coherent behaviour.

Actions imply interactions. Therefore, when learning is arising through the interactions it is possible to observe coherent behaviour where the changes in the actions of a person are in line with maintaining what he/she is doing. If the action is not coherent, the person will need another type of action in order to maintain his/her action in the environment.

An observer can note that there is a structural change in the action of the person, tracing the chain of action in the coherent behaviour of the person. That is learning (because learning happens in the interaction).

Let me explain with the next example. A student, for a while, might be adding integer numbers, say, $3 + 5 = 8$, $-3 + -5 = -8$. The strategy for solving these would be adding the numbers and the result will be another number with the same characteristic (a positive number or negative number). The action of adding this type of number works when both (or more) summands have the same sign before them or belong to the same category positive or negative number. This situation can be observed and specified as coherent behaviour (see chapter 3, section 3.9.4, p. 47) according to what is expected in the eyes of the observer (if the observer recognises the action of adding integer numbers, otherwise what is coherent for the observer cannot be the same as for another observer (see chapter 6, section 6.1, pp. 76-78).

If the addition of integer numbers with the same characteristic is working for the student, this means that he/she can still solve operations of addition with these numbers. The observer can note a coherent action and from that coherent action, the coherent learning of the person about addition.

However, what would happen when the next addition taking place is: -3 + 5 or -5 + 3? If the student is still with the coherent behaviour shown before, adding the numbers when they have the same characteristics (all the summands positive or negative), clearly what was working previously, the strategy, is not functioning anymore because it is not possible to apply that previous rule. The student needs to change his/her behaviour in order to continue operating in the mathematical domain of addition with integer numbers.

5.4 Metaobservation

The interactions amongst persons, in particular in this study interactions through conversations amongst a teacher and their students, are key elements to note how my learning emerges through the observations of the participants who are involved through their own actions. However, in order to observe the details of the actions, it seems to be necessary to establish the history of actions. From the observer perspective, noting if the person is still operating in the mathematical domain or, perhaps, he/she is moving on to another domain within the conversation taking place between the participants.

Finally, the distinctions noted in the changes of behaviour by the observer would show how those interacting make sense of what they are doing. In other words, how those operating in the same mathematical domain are showing a coherent behaviour in the eyes of the observer.

CHAPTER 6: Observing explicitly

6.0 Introduction

In chapter 5, I explained learning via interactions amongst persons or between a person and object noting that through a shift in the action of a person there is a distinction that could be seen by an observer.

In this chapter, I explain the observer role, his/her interaction with what has been observed. I specify how levels of categorisation (based on Rosch, 1978) made by an observer make explicit what has been observed through the act of observation.

I am presenting a theoretical framework that emerged through my own process of learning when I was studying the data collected (see chapter 7, observing and learning as research, section 7.4, p. 109; metaobservation, section 7.5, p. 110) based on awareness, behaviour changes, distinctions, empathy and emotion that allows me as observer to see the shift in actions of each person (in this study, the teacher and their students) when they are interacting in a mathematics classroom.

At the end of the chapter, I have written a metaobservation about observing using multiple perspectives (based on Reid, 1996; Reid & Mgombelo, 2015), including the use of categorisation and the making of distinctions as an observer.

6.1 Observation and the observer

What are we observing when we are observing an interaction? This seems like a trivial question, but in fact it is not.

From an enactivist perspective, what I am seeing in an observation is firmly linked to my own interaction of viewing the world. This interaction starts a loop between the observed environment and myself, triggering my own learning. As Bandura (1986, p. 51) says, “people cannot learn much by observation unless they attend to, and accurately perceive the relevant aspects of […] activities” - in other words, what is happening in the environment. This quotation, for me, is an invitation to consider the details between the observer and those who are being observed in the interactions within the environment where the observation is taking place, learning from those details.

Brown, J. (2017, p. 3), in a video-recorded episode from a mathematics lesson, illustrated that two observers in the same classroom are perhaps not observing the same differences in behaviour amongst students and therefore not the same structural changes in the action. In the account given by the author about his participants, he noted “some participants had their awareness drawn to the details of the task and to their own attempts to frame the activity in a context (mathematical/instructional)”. In this sense, the author provided evidence that it is possible to note how each action is dependent on every other, and in this action the specific way of knowing for each of the participants is shown such as mathematics content and instructions for the mathematical task or the framing of the activity.

From the observation, I need to make an explicit account of the situation observed (i.e., the students are working on Pythagoras's Theorem) but at the same time, explain the details of what I am observing, what I am taking account of in the interactions.

Mason (1987, p. 30), observing the difference between story and experience, pointed out:

> very often we move so quickly from fragment to story, *from an account of what we recall, to accounting for what we recall*, that we confuse the story for the experience [italics added in order to mark emphasis]

The observer perspective is *from an account of what we recall*, explaining what has been observed in the chain of interactions *to accounting for* the observation (the observer's own story).

Therefore, there is a constant loop between what has been observed (the account from the observer of the distinctions and categorisations that he/she makes) and the trajectory of the interactions that are being observed. There is a process for the observer of being involved in the observing and adapting to the environment, which triggers actions with others that would "modify the environment and/or the relationship of the organism to its environment" (Stewart, 2010, p. 3). The observer is part of the environment in which he/she is also interacting with the participants. As an enactivist, the observer cannot be 'external' to the environment. However, this does not mean I am taking part orally in any of the conversations between the teacher and their students. I am referring to being there, observing with internal decisions, whether conscious or unconscious (Brown & Reid, 2006) about what is being observed in the interaction. Thus, in this context, how can interactions be observed? What kinds of distinctions or differentiations am I bringing to what am I observing in the mathematics conversations carried out by the teacher and their students?

6.1.1 Categorisation

When I am observing, hearing, touching, smelling and so on—using my senses in my contact with the world—I begin an action that is translated into being and involves living in that moment. I start to select words, photos and textures, for instance, to make sense of what I am living. Categorisation is understood to stem from thoughts, perceptions, actions and speech (Lakoff, 1987, p. 5). From this perspective, categorisation is a natural part of us as human beings, it is what we do.

In this context, I start to categorise according to what I am doing while also categorising how I am doing what I am doing. If I am touching peppermint and lemon leaves, I might note that the lemon leaf is softer than the peppermint leaf, which is recognised through the distinction that I make regarding the texture of its surface. This would allow me to categorise these two leaves according to their characteristic of being either soft or rough.

Rosch (1978) proposed two general principles for the formation of categories by persons. Cognitive economy encompasses a great deal of information about the environment to categorise; it specifies the reason why a stimulus belongs to this categorisation and others do not (for example, a leaf belongs to the soft and not the rough characterisation).

The second principle considers how we perceive the world's structure and the attributes that we associate with the object. For example, for the peppermint and lemon leaves, I proposed a categorisation that was associated with their texture after I touched them (an action that can be seen by the observer). However, another person might categorise them through the action of smelling the leaves, which might produce 'citrus' and 'fresh' scents respectively from the lemon and mint leaves.

Therefore, "what attributes will be perceived given the ability to perceive them is undoubtedly determined by many factors having to do with the functional needs of the knower interacting with the physical and social environment" (Rosch, 1978, p. 29); an action could be triggered by contact with the world (structural coupling), and its categorisation is according to those actions (i.e., smelling and touching as basic actions in the leaf example) and the decision made in the moment by the knower; there are undoubtedly more actions performed by each person when distinctions emerge, such as perceiving at the moment of doing.

Naturally, what allows the configuring of the relations between a person and the categorisation is the constant loop of actions (the action performed to categorise, i.e., smelling, hearing, literally using the body to do it) between a person (knower) and what he/she is doing in the environment.

In the example above about peppermint and lemon leaves, the relation that I proposed was associated with their texture through the action of touching. In a mathematics context, sometimes primary students could be encouraged to follow the borders of geometric shapes to determine what figures are involved. Again, through the action that each person performs, the configuration can be established and perhaps named as the geometric figure, square (a form of categorisation).

In order to make explicit what has been categorised through action, Rosch (1978) and Varela, Thompson and Rosch (1993) point out three levels of categorisation: Subordinate, basic and superordinate (see table 1 in section 6.1.2).

6.1.2 Categorisation levels

According to Rosch (1978), common attributes can be found at the moment of interaction to determine a categorisation of an object by emphasising ways to perceive it that are inseparable from the way humans habitually either use or interact with this object (pp. 32–33).

These attributes can be part of three levels, as described in the table below:

	Subordinate level	**Basic level**	**Superordinate level**
Definition (based on Rosch, 1978)	A specific characteristic or attribute of what has been categorised	A characteristic or attribute of what has been categorised to make this object 'real' in the interaction	A general characterisation of the objects' categorisation
Example object	e.g., kitchen chair, desk chair	e.g., chair	e.g., furniture

Table 1: Categorisation levels (Rosch 1978, p. 32).

Varela, Thompson and Rosch (1993, p. 177), based on Rosch's work, note that "The object appears to the perceiver as affording certain kinds of interactions, and the perceiver uses the objects with his body and mind in the afforded manner". A person can note the object, because of interacting with it, for example, noting the floor (conscious or not) because she/he is standing on it. If the person perhaps replies, "I am not touching the floor", it might be because they are floating in water. They cannot answer the question through no interaction with the object floor. The name 'floor' appears to the person through the interaction that she/he has with this, in this case, standing up.

Standing up is a related action for the basic level category floor. Rosch *et al.* found the basic level of categorisation [for example, chair or floor] "to be the most inclusive level at which category members (1) are used, or interacted with, by similar motor actions, (2) have similar perceived shapes and can be imaged, (3) have identifiable humanly meaningful attributes, (4) are categorized by young children, and (5) have linguistic primacy (in several senses)" (Varela, Thompson & Rosch, 1993, p. 177).

In this context of categorisation, the "researcher/observer" (Maheux & Proulx, 2015, p. 215) plays a crucial role. The observer (in this case me as a researcher) distinguishes what has been observed in the action that determines the categorisation level. In addition, the action of the observer would also be categorised under Rosch's basic level criteria, if we consider that the observer also establishes relationship through the action of observing the interactions of others.

Naturally, any distinction from the observer is not unique and is dependent upon the relation that the observer can establish or not with what he/she has observed, because the observer is also interacting with the actions observed and making categorisations as well. For that reason, it is crucially important, to me, what I have discussed in section 1 in this chapter about making explicit what has been observed in the interaction of the others.

Returning to the example described previously, an example of categorising an object in a mathematics context would be through observing and touching a shape with four interconnecting sides. At a subordinate level, an observer could describe it as a figure with four sides and four connecting angles; at a basic level, the observer could observe and touch the four sides of the figure, noting that it is a square, because of, for example, fitting with other squares, creating a tessellation (their interaction with the object), seeing the square, while the observer at the superordinate level reports a quadrilateral.

What is noted in the observation of the shape will be dependent on the relationship; this means the interaction that each person has with the shape. However, considered from the perspective of the observer, the levels proposed by Rosch (1978) offer a characterisation to categorise the action from the specific characteristic (subordinate level) to the general (superordinate level).

The example provided above could be extended, considering an interaction that involves mathematical conversations in a study made by Chan, Wan, and Clarke (2018, pp. 227–229) about social interactions when 13-year-old students were solving three different mathematical tasks in pairs. One of the tasks was:

> The average age of five people living in a house is 25. One of the five people is a Year 7 student. What are the ages of the other four people and how are the five people in the house related? Write a paragraph explaining your answer. (p. 227)

A conversation follows between two students:

> Ji-na: So we need to get 125.
>
> Nafisa: Oh that's going to be . . .
>
> Ji-na: The age. So . . .
>
> Nafisa: Okay. So, that − it's simple. Okay. So 25 − I have a very good one. Wait, wait, wait. Twenty-five, 25 equals 50, so one more 25 equals . . .
>
> Ji-na: So if we put . . .
>
> Nafisa: Wait, wait, wait. I − I have it, I have it, I have it, I have it, I have it. Yes. That will work, I think. So 28, one, two, three, four, one, two, three, four. So 4, 32, 4, 7, 72. No. That doesn't add up. . . . why can't they have 25, 25, 25, 25?

Although the researchers were noting how the students solved this problem, I am interested to show how this transcript could be observed using a categorisation approach.

If I observe the transcript above categorising from the subordinate level, this means looking for a specific characteristic in the conversation between these students, which could be looking for the ages (specifically, in this conversation, a type of interaction). What is common? They are doing the opposite of averaging, trying to find the ages of five people living in the house when the average is 25.

The basic level, which is a characteristic that makes the object real in the interaction, for me as an observer would be obtaining the average of different ages. On the one hand, Ji-na is highlighting, through 'playing' with the age numbers, how they can find the average of 25, whilst Nafisa, on the other hand, is literally interacting (or playing) with the numbers in order to find the average of 25.

Finally, the superordinate level is a way of categorising how to observe the whole, ‘conceptualising’. In this sense, what is brought forth is a general description from the observer that would include the type of dialogue that triggered the interaction, such as, a discussion about ages and averages in the transcript above, which enables a continuing dialogue.

From my point of view, the levels of categorisation do not work in a hierarchical order; on the contrary, they work in a complementary fashion, as the example above illustrates: Looking for ages, averaging with different ages, and talking about ages and averages. This complementary way of working allows the observer to note what happens in the interaction in greater detail than staying with one description.

Therefore, levels of categorisation based on the attributes of an object seem to be firmly related to distinctions that I can note through the observations at the moment of interaction with that object, such as described in the example about ages and averages when two students are working on the mathematical task.

6.1.3 Using categorisation in research

Different researchers have used the levels of categorisation mentioned above to correlate their relationship with the kind of action performed (i.e., show our behaviour). Brown and Reid (2006, p. 181) noted that “basic-level categories helped them deal with complexity through pattern recognition” when interactions took place between participants, creating an explicitly conscious process of the action through the decisions that students make in mathematics.

Slingerland (2008) associated the basic level as follows: "[T]he first categories learned by children are shared cross-culturally and are experienced as being easier to remember and more conceptually important than more detailed or more general categories" (p. 121). First categories specify for him a way of categorising language learning, i.e., if they are learning a new language, it is better to start with the word table, chair or bench instead of expressing how a kitchen is composed, which involves furniture.

Although Slingerland (2008) did not mention behaviours and actions by children, he highlighted experience in a cross-cultural environment. I believe that experience comes from doing and performing a kind of behaviour that relates objects to us. Notably, the object that we call a chair is a categorisation of the function of our doing with the object such as sitting.

Finally, to interpret the conversation between mathematics teachers when they were working on recorded videos, Coles (2011) linked these levels of categorisation with three layers of abstraction. In other words, detailed layer as an account *of;* basic-level categorisation was a 'direct' way of categorisation rather that the superordinate level, which involves generalisation of the action.

The details observed in the action, through the categorisation made, can offer an account of how the participant could arrive at new possibilities for action, showing their autonomy through the decisions made.

6.2 Awareness and behaviour changes

In the context of a classroom, and specifically a classroom where the students and teacher are doing mathematics, many interactions are happening. Some of the interactions are perhaps not directedly related to a mathematics concept. For example, the teacher asking students to open a textbook could certainly be supporting learning behaviour. If I open the textbook, I would be looking for something, maybe the same as asked for by the teacher, or maybe something that starts to attract my attention in the textbook, for instance, other questions written there, or a particular word, or a mathematical figure.

On the other hand, from a question or any perturbation (an action) surrounding a mathematical action from teacher or students, it is possible to observe a network. This is a configuration that emerges through a chain of actions from the participants with their surroundings. The formation of the configuration is not from cause and effect because from a biological point of view our way to act is not in one direction.

An example of this lack of cause and effect is illustrated by Varela's work and the way our brain works, which is named by him the *Law of reciprocity*: "if a region (say a cortical area, or a specific nucleus) A is connected to another region B, then B is also connected to A, *but by a different anatomical route*" (Varela, 1999, pp. 46–47, emphasis in the original) i.e., going from A to B in a conversation between two students will be not the same as going from B to A to others. Instead, each person can show his or her particular way of interacting, and particular way of being aware of an 'event' (as happens with Ji-na and Nafisa, shown in section 6.1.2: Nafisa was aware of using different ages in order to find the average of 25 and Ji-na was aware about what the problem said), which enables the observation of the change in their behaviour.

From my enactivist position, it is possible to distinguish different actions of knowing, even though there are many interactions happening. Within these interactions, one possibility is noting how the action of each one is in line with his/her awareness. Gattegno said, "knowing [an action] is the awareness that one is aware of something and according to whether we stress the something or the awareness, we progress in the subject or in the education of our awareness" (1987, p. 40). So, it seems to be that awareness is part of our action of knowing, making conscious the focus of our actions.

In this context, how can the action of awareness be recognised? Hewitt (2001) proposes that a teacher should be aware of working with the arbitrary and necessary in mathematics. For him, the arbitrary is liked with memorising (i.e., memorise the name fraction, or cultural mathematics conventions), however, memorising something in mathematics is not doing mathematics. The necessary is linked with working with the awareness of the students, that is to say, being aware of the conditions of the problem that make it realistic, for example, we can agree that we cannot have an answer of a negative age. However, he recognises that "if something is necessary, this does not imply that everyone can know it through their own awareness" (p. 44).

What kind of perturbation can trigger awareness of action in each person? How can the awareness of action be recognised from the observer role?

Let me explain with the next example. Mason (2005, p. 2) illustrated a situation that he named “smiley face”. He was observing a mathematics teacher working through a bracket expansion of a mathematical expression such as (x + a)(x + b). As he observed, the teacher reminded the students of the “the device of linking all pairs of terms to be multiplied by arcs, producing two eyebrows, nose and chin, with the outside brackets as ears”, as shown in figure 13.

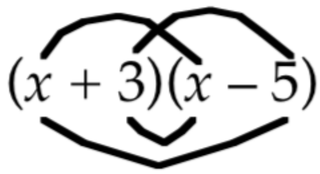

Figure 13: Smiley face (Mason, 2005, p. 2).

In this case, in order to trigger the effective action of the multiplication of the algebraic terms from the teacher to their students, the “smiley face” perturbation was used. This perturbation perhaps allows the students to be aware of the type of multiplication carried out; for example, the student can ask, “Do I have the “smiley face”?” However, is it enough to have the “smiley face” to demonstrate mathematical awareness about the algebraic multiplication of an expression, such as (x + a)(x + b)?

What happens if the student has the awareness of the “smiley face”, but even then, when she/he performs the action of multiplying each term a wrong answer results, such as, let us say, in the example in figure 12, $x^2 – 15$?

At this point, I can recognise that knowing as an action is making explicit our awareness in the action. Nevertheless, “our consciousnesses [does] not direct our thoughts and actions” (Davis, 2004, p. 178). These are guided by our perceptions from living in the world. To multiply correctly all the pairs of terms in the algebraic expression, something that is necessary to educate (Hewitt, 2001, p. 44), we might be aware of the "smiley face" but for me this does not mean our action will be directed to a correct procedure.

However, "consciousness does play an important role in orienting attention-that is, through differential attention, in selecting among the options for action" (Davis, 2004, p. 178). Orienting attention, in the "smiley face" example, is necessary for expanding the terms in the multiplication of expressions such as $(x + \mathrm{a})(x + \mathrm{b})$.

In the selection of future options, we show our autonomy through the decisions made in the environment but also through our awareness into action. How can the awareness be observed in the action of each person or between persons?

Brown and Reid (2006) reported that somatic markers of unconscious activity can be researched through the decisions (actions) made by both teachers and students in mathematics classrooms. Similarly, Varela (2000) noted that the core of becoming aware in living experience is associated with the basic process cycle proposed by Depraz, Varela, and Vermersch (2000, p. 42), which is called *epochè.*

What is common in these two approaches is that both are observing a shift in action from the participants in their interaction with the world. This means that the participants act differently and that action can be noted from the observer's perspective in the historicity of interactions of each person with their environment.

In my view, I need to establish the chain of interactions of each person in order to note as an observer the moment of being aware of something mathematically. Otherwise, there could be an action, as in the example described, that is of the awareness of the “smiley face” without there being awareness of the process of multiplication of algebraic expressions.

Naturally, ‘awareness of’ depends on each person and her/his relation to what he/she is doing (see chapter 3, sections 3.9.2, p. 45 and 3.9.3, pp. 46-47).

In this context, when a person is interacting with others, from the observer’s perspective, certain structural changes linked with learning (see chapter 5, section 5.2, pp. 67-71) can be noted in the action of what the observed person is doing.

This structural change can be observed from different perspectives dependent on the focus of the observer. What is the awareness of each person who is interacting with the observed?

For example, one of the usual conversations between students within a mathematics classroom, after doing some work, is about what the result is of the operation.

Let me suppose there are two students in a mathematics classroom working on solving addition with fractions with different denominators such as $\frac{3}{4}+\frac{1}{2}$. The next dialogue follows after one of them asks about the result of that addition. S1: Student 1, S2: Student 2.

1	S1:	What did you get?
2	S2:	Let me see 4/6
3	S1:	I got 5/4
4	S2:	How is that?
5	S1:	Because a half is the same as 2/4 so I changed 2/4 instead of ½ so it is 5/4.
6	S2:	But, that is not right, it is more simple, it is just adding the numbers, three plus one and four plus two. Did you see it? Like the usual numbers, two plus two is equal four
7	S1:	So, you are saying that just adding the number without taking account of the fraction?
8	S2:	Do you mean three divided by four?
9	S1:	Yes, or one divided by two. Did you get it?
10	S2:	Oh, I see it. I can't just add as three plus one. I must do something more like you did it

In the dialogue above, although it is an unreal situation, I can say that there are two students speaking about the results, with two different approaches of behaviour to solving the addition of fractions. Student 1 (S1) is speaking about adding making a change in one of the fractions, see line 5 "half is the same as 2/4", whereas student 2 (S2) is adding the fractions by adding the numerator and the denominator respectively as shown in line 6, "it is just adding the numbers, three plus one and four plus two".

In line 10, student S2 becomes aware that she/he cannot do adding fractions by adding numerator and denominator, as shown in the expression, "Oh, I see it. I can't just add as three plus one. I must do something more like you did it". This can be considered to be a structural change, acting differently compared with the first action on line 6, when she/he explains the procedure to S1 about adding fractions as "usual numbers".

To note the actions of becoming aware, in the current study I will consider the *epochè* cycle and the description given by the authors. The *epochè* cycle consists of three actions that work together simultaneously—suspension, redirection and letting go—in the process of becoming aware (figure 14).

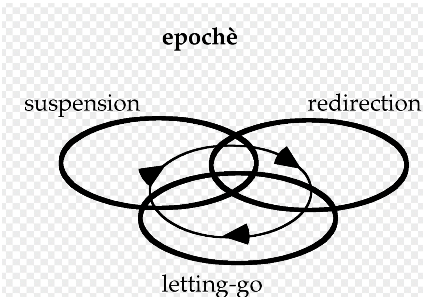

Figure 14: The epochè cycle (Depraz, Varela, and Vermersch, 2000, p. 25).

Suspension: *"[S]uspending* your "realist" prejudice that what appears to you is truly the state of the world" (Depraz *et al.,* 2000, p. 25, emphasis in the original). It becomes necessary to remove ourselves from our usual actions; we must move away from our usual way of engaging in an activity and create space for new experiences (Varela, 2000, p. 4). Based on what these authors proposed, from my view, it is a moment to re-direct our self from what we were doing previously.

Redirection: "*Redirecting your attention* from the "exterior" to the "interior" (Depraz *et al.,* 2000, p. 25, emphasis in the original) via suspension, which leads to emerging events, content, patterns, gestures, etc., from whence we can find the new view (Varela, 2000, p. 5). In this context, redirection is an action that allows us as persons to move on from habits with which we are confronted by suspending ourselves and move into a different action, recognising what we have done but noting, at the same time, the 'something' new.

Letting go: "*Or accepting your experience*" (Depraz *et al.,* 2000, p. 25, emphasis in the original). This action is defined by acceptance, which is triggered by the quality of the attention. In this sense, a person is actively paying attention while reflecting. From here, the self is actively balancing between attention and lacking fulfilment (Depraz *et al.,* 2000, p. 47). From my view, this is a moment of decision where the person can follow the same action in which the interactions started or move on to a new action, accepting 'the new'.

To illustrate these three moments: suspension, redirection and letting go, named *epochè*, in a conversation, let me return to the example presented at the beginning of this section. I will repeat the transcript in order to facilitate the reading.

1	S1:	What did you get?
2	S2:	Let me see 4/6
3	S1:	I got 5/4
4	S2:	How is that?
5	S1:	Because a half is the same as 2/4 so I changed 2/4 instead of ½ so it is 5/4.
6	S2:	But, that is not right, it is more simple, it is just adding the numbers, three plus one and four plus two. Did you see it? Like the usual numbers, two plus two is equal four
7	S1:	So, you are saying that just adding the number without taking account of the fraction?
8	S2:	Do you mean three divided by four?
9	S1:	Yes, or one divided by two. Did you get it?
10	S2:	Oh, I see it. I can't just add as three plus one. I must do something more like you did it

As I showed previously, there is a change in the behaviour of student 2 about how to add the fractions mentioned. How can this change through interaction be observed under the *epochè* framework?

In line 4, with the question “How is that?”, S2 is opening their mind to possibilities, avoiding prejudice of their current knowing of adding numerator with numerator and denominator with denominator. Suspending his/her self, S2 is moving their attention to what is really happening with actions with others. Then there is a redirection of the action as shown on line 8, noting the fraction as a division with the question, “Do you mean three divided by four?” and not an integer number. On line 10, there is a moment of acceptance when S2 says, “Oh, I see it”, letting go (of what she/he has done previously) with, “I can’t just add as three plus one”, which focuses attention on the action of the decision when S2 notes, “I must do something more like you did it”.

It becomes possible for the observer to see that each characteristic is not working separately; rather, each supports the others without a hierarchical order, because the ‘order’ is given by the observation made by the observer as shown in figure 13.

I described in the example that there are two students with two different approaches to adding fractions. Each moment of awareness includes characteristics that are inherent in each person and their unique way of acting within the world i.e., S1 in the example described is associated with structural coupling since S1 is ‘part of the world’ of S2 and vice versa. Structural determinism is their unique way to act and take their decisions according to their own autonomy.

These characteristics of becoming aware would be visualised through the eyes of the observer regarding shifts of action when a person follows their own pathway, such as when two people are interacting in a conversation that can involve working with something new to them or remembering facts from the past that they have already experienced.

In this sense, the process of becoming aware is personal; no one can claim to be ‘in the shoes of the another’ because that is impossible, especially considering that our individual structure makes us who we are. The observer can note that the changes were triggered by the action, as shown by Reid and Brown (2006), regarding the unconscious process that results from decisions in the moment of letting go.

In addition, the observer can note changes in the interaction through conversations/dialogues that were previously established, as described by Depraz *et al*. (2000).

The moment of becoming aware can be related with the action of what a person is doing at the moment of interaction with others/objects. However, each person can be either aware or not in that moment because, as Davis noted, “our consciousness does not direct our thoughts and action… but plays an important role in orienting our attention in our actions” (Davis, 2004, p. 178) triggering acts of knowing for them.

6.3 Distinctions

One of the basic acts of cognition, (mentioned in chapter 3, section 3.8) is the act of distinction, or in Maturana and Varela’s (1992) words, “the act of indicating any being, object, thing […] involves *making an act of distinction* which distinguishes what has been indicated as separate from its background” (p. 40 emphasis in the original).

As an observer, to indicate what has been separated from the background, I need to note ‘something’ in the pathways of actions, otherwise how can I distinguish there is ‘something’ perhaps different in what I am seeing (my doing) currently?

We start facing a situation (in which obviously we are involved through our structural coupling with the world), establishing references for what we are observing in the action, meaning that “we are specifying criterion, which indicates what are we talking about” (Maturana *et al.,* 1992, p. 40).

Watson and Mason (2005, p. 154) mention exploring distinctions because they are at

> the heart of most mathematics and most learning concepts. Creating objects that explore the boundaries of definitions is a powerful way to learn about mathematics structures.

They propose a ‘reversal’ idea of working with mathematical distinctions. For example, instead of solving an equation, give the solutions first and ask the students, “What could the equation be?” (Watson & Mason, 2005, p. 155).

Within a mathematics classroom there are many distinctions that could be made from interactions that happen in that context. For example, a distinction of mathematical knowledge, i.e., say students are working in geometry instead of algebra; a distinction of the relationship of doing mathematics, i.e., say students like or do not like mathematics. What is common, in these examples, is to make a distinction and specify what we are talking about. Again, the role of the observer is presented, not only in the person who is performing the act of distinction in his/her own act but at the same time in the observer who is observing the action of the others in their pathway.

6.4 Empathy and emotion[6]

Students' emotions regarding mathematics have been widely researched from different educational approaches. A review of different perspectives is presented by Zan, Brown, Evans, and Hannula (2006).

Hannula (2012) drew a theoretical framework related to the affect of mathematics, including how it is related to cognitive, motivational and emotional aspects of affect. Other authors considered emotions related to students' interest in a mathematical task, which could be manifested through their engagement and affected by the teacher's role in that engagement (Nyman, 2017). For example, a study with German students (aged 12–13 years) about interest in mathematics showed it could be considered to be a predictor for mathematics achievement (Heinze, Reiss, & Rudolph, 2005, p. 212).

Reid and Drodge (2000) argued that, "emotions play a positive and central role in mathematics" (p. 249) when students and teachers are doing mathematics in the classroom, showing that, according to Reid and Drodge, they are speaking of

[6]The literature reviewed about emotion and empathy is part of my paper, Empathy in interactions in a grade eight mathematics classroom in Chile. In J. Golding, N. Bretscher, C. Crisan, E. Geraniou, J. Hodgen & C. Morgan (Eds.), *Research Proceedings of the 9th British Congress on Mathematics Education*, (pp. 143-150).

emotional orientations or shared preferences (p. 249). Another author who discussed positive emotions, Ramirez, M. (2005, p. 109), suggested that Chilean grade eight (13-14-year old) students "have very positive views of mathematics: three-quarters report liking or enjoying this subject, and more than half would like a job involving mathematics in the future". From the point of view of the emotions, mathematics is perceived by these students (13-14 years old) as "enjoying".

Despite the large number of studies on emotion in mathematics, and taking into account that "emotion is the origin of what we do every day, in our doing and interaction with the world" (Varela, 2000, p. 247, translated from Spanish), there is little focus in the literature reviewed on how emotion is manifested through empathy in the interaction between a teacher and their students in a mathematics classroom. Empathy arose for me through the observation of the data collected (see chapter 8, pp. 134-137; chapter 9, pp. 152-188; chapter 10, section 10.3.6.1, pp. 235-240).

From the point of view of interaction, I am considering that "biologically, emotions are bodily dispositions that determine or specify a domain of actions" (Maturana, 2001, p. 8, translated from Spanish). For example, if I observe two persons in the street and one of them starts smiling and laughing, giving a hug to the other person, how can I note that they are happy? Or, how can I observe the emotion named happiness. If I observe the type of action that they are making with their body, perhaps I can hear they are laughing. The sets of actions that entail a specific way to act, for me, is the domain of actions that can be seen as 'happiness' (for more discussion about happiness see chapter 5, section 5.1.1, pp. 66-67).

In a similar way, there is a domain of actions in which empathy is taking place between the persons who are doing the actions.

Lord-Kambitsch (2014) says the meaning of empathy is ambiguous, because it "differs according to specific context" (p. 4). In the everyday actions of human beings, including those interactions that may happen in a mathematics classroom, empathy could be characterised with a "special status" that is distinct from every type of social cognition (Kaplan & Lacoboni, 2006, p. 355).

Cooper (2004) distinguished empathy in teaching and learning, such as being interested in facial expressions, language and tone of voice into *functional empathy* within a group, which can include discipline and relations amongst the group; and *profound empathy* as the development of positive emotions and interactions, such as acting and taking responsibility; adaptive and integrated concepts; and moral aspects (p. 15). From my enactivist point of view, the description of empathy given by Cooper suggests actions performed by students or the teacher when their interaction in the classroom takes place.

Other researchers have considered the importance of empathy to be "a characteristic of a teacher that enables adequate communication between the participants of the educational process" (Stojiljkovic, Djigic, & Zlatkvic, 2012, p. 961). Or, seeing this characterisation from an enactivist point of view, how the teacher's actions can be triggered or not to create adequate communication with their students.

Gallagher (2012) noted that empathy can (by default) be driven by the neuroscience of mirror neurons, which "provide a mechanism by which we can understand the actions of others by mapping the actions of the other people onto our own motor system" (Kaplan & Lacoboni, 2006, p. 175).

In this thesis, and inspired by the mirror neurons idea mentioned by Gallagher (2012), I will consider there to be empathy in the interaction between teacher and their students when "there is recognition of the other's experience as belonging to the other, without losing the distinction between self and other" (Thompson, 2001, p. 6). That is, I project myself in the other, but I am still being me, i.e., when certain interactions happen between teacher and their students, the teacher can project an action led by empathy from the mathematical action of their student.

In this recognition of the other as 'part of me', Depraz and Cosmelli (2003, pp. 171-172) consider the next stages of empathy to be:

1. a passive association of my lived body with your lived body.
2. an imaginative self-transposal in your psychic states.
3. an interpretative understanding of yourself as being alien to me.

Number one refers to our structural coupling with the world; this is "foremost conditioned by my visual perception of the body of the other, which would mean that we mostly have to do with the meeting of two perceptual and reflecting body images" (ibid., p. 172). This can be noted, for example, when someone raises his/her hand to say hello and the other person simultaneously makes the same gesture.

The second point includes "recalling a similar experience, where I had such mental states and I am then able to feel empathy" (ibid., p. 172); therefore, we can 'see' in our body perceiving the other as allied to us. For example, when someone tells a story and smiles, and another person notes a similarity and smiles as well.

Number three "involves expression (verbal or not) and interpretation, which lead to the possibility of understanding (and misunderstanding, of course): it is a cognitive step" (ibid., p. 173).

To specify this recognition of how empathy can be observed through interaction in conversation between a teacher and their students, in my analysis (see chapter 10, section 10.3.6, pp. 235-240) I will consider the third complementary stage of empathy, identified by Depraz and Cosmelli (2003).

6.5 Metaobservation

What I am noting in any observation will be dependent on my relationship with the world, the interaction that I am doing. Obviously, there is a goal to observe but for me the goal acts as a facilitator and not as instructor. My goal is to observe the emergence of mathematics learning, but with this goal I am looking to pay attention to my thoughts, not to generate judgmental actions.

The use of categorisation to take account of the level of observation, based on Rosch (1978), allows me to make explicit what I will observe as a researcher in the collected data (from the field notes and video-recording of the same mathematics lesson).

What I am re-calling in the observations of my fieldwork after the event is an account of the categorisation that I am doing as a researcher. Categorising implies unique actions from me that are according to my own structure and autonomy expressed in the decision taken.

In addition, categorising implies actions of making distinctions. Distinctions that I will make through observing the distinctions made by the teacher or their students and how these can be linked (or not) with other categorisations of actions (awareness and behaviour changes as well as empathy through the interactions).

What is common between these levels of categorisation? (making distinctions and unique actions to categorise)? Both share the inter-actions of the participants therefore it is possible to observe the actions of them. I am using inter-actions, because I would like to stress the inter actions that imply the interaction. Observing the action of a participant and the inter-action of the participants, in this study teacher and their students, allows me to observe the changes of behaviour when the interaction takes place, for example in a conversation.

By considering multiple perspectives, different approaches that could be explained (Reid, 1996; Reid & Mgombelo, 2015), came richness. For me, using these levels of categorisation allowed me as a researcher to use different types of lenses to observe in my interaction with my study.

How can I explore my own interaction with my study without focusing my goal as a researcher? In chapter 7, the next chapter, I will discuss enactivism as a methodology, which for me pointed out the learning that happens in the interactions of a researcher with the research project.

CHAPTER 7: Enactivism as a methodology

7.0 Introduction

In part one, I have described my experience, noting how my epistemological position moved to an enactivist view.

One of the main concerns in this thesis is the relationship of the teacher and the students in a mathematics classroom when the learning emerges through the interaction (see chapter 5), specifically through conversations about mathematics.

In this chapter, I present and describe my methodological position, enactivism, and also how my decisions are shaped by this position. As a researcher, I am working with 23 students and a mathematics teacher in an 8th-grade (13-14-year olds) classroom in a Chilean school.

I show how my interactions with this research project, in my positions as observer and learner in this research, involve acting as a researcher in circularity with the environment.

I conclude with a metaobservation, noting that enactivism as a methodology, “learning about learning” Reid (1996, p. 205), can be seen as interaction about interaction.

7.1 Methodology

It is important to consider what my methodology is, the position within my research that I have adopted to show what decisions I have taken within the period of doing my research. This chapter gives an account to the reader about my study but also shows how this position shaped my way of interacting (and therefore learning, see chapter 3, section 3.3, pp. 27-29) with the participants whilst the study was being carried out. For example, as shown in chapter 8, section 8.8.3 (pp.137-140), working gradually and iteratively in the process of collecting data.

According to Crotty (1998, p. 3), a methodology provides a context for the investigation and grounds it in a particular research paradigm. For me, it is a tool to understand what I am performing as the research and how I am performing the research. There are different approaches to research that depend on the nature of the study, its objectives and the philosophical position of the research. Briefly, for example, some studies "rule […] out alternative explanations of findings deriving from it" (Bryman, 2008, p. 693). Therefore, the role of such research is independent of reality, because the findings come from a 'rule', such as, using a questionnaire, working with different trials where the findings looked for from the participants of the study are in the outcomes of the questionnaire, without taking account of the obvious influences on outcomes of the participation of the researcher and the participants. As Varela *et al*., (1993, pp.13-14) noted,

> To deny the truth of our own experience in the scientific study of ourselves is not only unsatisfactory; it is to render the scientific study of ourselves without a subject matter […] Experience and scientific understanding are like two legs without which we cannot walk.

How can I take a parallel position in my study? Reality is not external; on the contrary, I am living in that moment and therefore learning as well. Therefore, it is necessary to express what has been my position in this research as a researcher who was interacting and learning in this process.

Considering that the nature of my study is re-observing (my interaction as a researcher) the emergence of learning in the interactions between a teacher and their students and between the students themselves under an enactivist perspective, one of the major characteristics is that cognition is an act of learning that happens in the interaction with others (see chapter 3, section 3.3, pp. 27-29).

I am observing this world, while at the same time I am acting in it, there is not a division of me and the world, I am part of the world and the world is part of me (based on structural coupling, chapter 3, section 3.9.2, p. 45) doing a constant loop of what I am seeing and what I am distinguishing; therefore, I continue learning all the time. In this sense, considering my perspective on the world as an act of doing, I cannot be separated from my research because I am also part of the study with a different role, a researcher role.

Consequently, the methodology for my research will be enactivist because it is a theory of "learning about learning" (Reid, 1996, p. 205).

7.2 Enactivism as a methodology

"learning about learning" (Reid, 1996, p. 205)

Enactivism as a methodology in mathematics education research is a theory for learning about learning (Reid, 1996, p. 205) because learning is a phenomenon of a particular moment of experience in a particular context (based on Stewart, 2010, p. 9). In other words, one's experience with the environment or other organisms offers the opportunity to do an action within or with them, learning. I am learning about the emergence of mathematics in the interaction between a teacher and their students and between the students themselves.

Learning with the methodology of enactivism works through multiple perspectives, which could be through the participation of multiple researchers; multiple revisitations of data; and the act of communicating our research to others (Reid, 1996, p. 207).

At the same time, "enactivism is concerned with the [sic] *learning in action*" (Khan, Francis, & Davis, 2015, p. 272, emphasis in original). It does not mean constructing ideas according to the theory of constructivism; on the contrary, the focus is on the cognition of each individual and his or her relationship with the environment. Therefore, as a researcher, I cannot separate myself from the context or take an objectivist position, such as "that the things exist as meaningful entities independently of consciousness and experience" (Crotty, 1998, p. 5). I am going to live the experience with my participants through the research. The difference is that we have different roles, but we will be situated in the same mathematics lesson. For example, I am observing the actions that occurred, when my participants were doing and learning mathematics. I will try to describe a history of the interaction that makes sense to me of the things that are happening in there. I am learning how aspects of mathematics can emerge from the cognition of the participants through his/her interactions with others in a mathematics conversation. “Mathematical cognition is observed as a doubly embodied ongoing action in an environment” (Kieren, n.d., para. 3). This means that the structure of each person, the teacher, each student or myself in this study, determines the way that he/she will act within the environment, with the environment providing the space in which to act (based on Kieren, n.d., para 3).

Each mathematical ‘idea’ from the student or teacher is carried out with action in the environment. As a result, each structural change means acting differently (as I discussed in chapter 5, section 5.1.1, p. 65). Each participant in this study shows me ways of adapting a mathematical situation, doing a pathway within the mathematics lesson linked to what is the meaning for them, acting coherently on the pathway that they are doing and with the consideration that I mentioned in my research question.

The pathways can be observed by an observer, (see chapter 7, section 7.4, pp.109-110, for more explanation about observing) because “enactivism exists in the eyes of observers, and so any discussion of it must begin with a description of the authors’ observations” (Reid, 2014, p.138).

Consequent to this, as a researcher I am trying to bring the reader and the scientific community the details of what I am observing and how I am observing this aspect and not others. My interpretation of what I observe should be in step with my description and my observation, as Reid and Mgombelo noted, “the observers and interpretations must be mutually explicable" (2015, p. 180) with the purpose that the audience can read what I have seen.

7.3 Acting in circularity with the environment

In the light of the worldview that Varela, Thompson, and Rosh (1993) noted, "[w]e reflect on a world that is not made, but found, and yet it is also our structure that enables us to reflect upon this world" (p. 3), there is a strong relationship between each person and the environment.

As a researcher, I am distinguishing to make sense of the behaviours observed in the interactions between my participants. I am also distinguishing my own distinctions that affect my own behaviour so that I am able to see these actions of the students and teacher; it is a perfect ensemble of two sides working hand in hand.

The state described above is recognised as circularity, an interdependency between our structure and cognition as an action; that is to say, taking account of the background and the biological, social and cultural beliefs of the moment to interact with scientific understanding (c.f., Varela, Thompson, & Rosch, 1993, chapter 1).

This circularity is important to me because, as researchers and simply because we are human beings, we are living serving as a constant compass between our structure and the experience of our interactions with the world which also provide the space to interact with them. The students and teacher are living in a mathematics classroom, which at the same time also provides the space to interact because the classroom is the teacher and their students.

7.4 Observing and learning as research

Based on my enactivist approach, I would like to broaden my thinking about observation. I am living and interacting in the 'world' observed so what can I see in the interactions performed by others? There is not an established distance in the world between what has been observed (i.e., the participants in this research) and the observer because both, participants and researcher, are present together.

At the same time these observations are addressed by my own structural determinism, my 'own sense' of seeing what I am seeing (chapter 6, section 6.1, pp. 76-78, *observation and observer*). As Maturana (1988, p. 3) noted,

> if someone claims to know algebra -that is, to be an algebraist- we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim

Consequently, I, as an observer observing somebody performing 'algebra', start to develop my own criteria based on what I am understanding and what I am learning in the selected observations about acting as algebraist.

Naturally, there is not only one criterion to select what I am observing, because, from the methodological perspective, being in this loop of learning through the observation, different observations (a type of interaction) can take place and therefore different learning arise through those. However, the criterion of selection should be according to coherent actions in my own historicity of learning in this process of observation (see chapter 9, *My Historicity,* pp.152-189). At the same time, there is a coherent behaviour in the actions observed from the participants, which would be noted by the observer and therefore a claim could be made such as, for example, he is acting as an ‘algebraist’.

7.5 Metaobservation

From an enactivist point of view, the environment entails opportunities to act but each person decides how to engage in those opportunities. When I am observing, using my senses in contact with a mathematics classroom, I begin an action that is translated into being and involves doing in that moment. I start to interact with the classroom, the first part of my learning. Later, I will interact with the data collected from the first part of my learning, which is also learning. When Reid (1996, p. 205) said that enactivism is a methodology of “learning about learning”, for me this is seen as interaction about interaction. From an enactivist point of view, learning emerges in the interaction that we have with the world and the world with us, without division between the world and us, because ‘the world’ is doing and doing is interacting.

I will show my own learning from an enactivist perspective (based on Reid, 1996) through the changes between the two instances of observations of my data. I will be moving my focus from the description level in the first observation to looking for greater details when I observe the data collected through video-taped lessons and audio-recordings. However, my learning is not only with the data collected, my learning also implies the constant process of the reflecting within my analysis and the writing up of this thesis.

PART THREE: Conversation about mathematics

CHAPTER 8: Context and method

8.0 Introduction

In part one, I have presented my experiences, concerns and dilemmas as a result of being involved in mathematics education, showing how I have come to work with my epistemological position, enactivist theory, and highlighting some concepts of the theory. Part two refers to learning and my methodological position enactivism, presenting a theoretical framework that I later use as a part of my view of learning mathematics.

In the first half of this chapter 8, I fill in some background of mathematical modelling and the Chilean curriculum. In the second part of this chapter, I describe my research design, explaining the decision that I took based on my methodological position.

I conclude this chapter with a metaobservation about my iterative process, the decisions taken through the interactions involved in the doing this research.

8.1 Mathematical modelling

Mathematical modelling works with a mathematical model to represent and solve a mathematical situation. Historically,

> Until the first third of the twentieth century most of the mathematical models were used to describe phenomena and to make qualitative statements about the real world problems described. Since then the

> situation has changed dramatically. The tremendous increase in computing power has shifted the interest in mathematical models more and more from problem *description* towards problem *solving* (Schichl, 2004, p. 31, emphasis in the original)

Frejd (2013) says that individuals who are doing mathematical modelling, or 'modellers', manifest the importance of qualitative data and validation, adding that communication between industry and the person who is doing the modelling is essential for a model to be successful.

For me, whether this final representation, the model, can be used to solve problems depends on the persons who are doing the modelling the problem solving and the applicability of the model's results.

Education's interest in mathematical modelling has been increasing, and this can be seen in the papers presented at international conferences and their working groups (i.e., International Congress on Mathematical Education (ICME); Conferences of European Society for Research in Mathematics Education (CERME)), however, "only about 3% of the contributions published in *Educational Studies in Mathematics, Journal for Research in Mathematics Education* or *ZDM Mathematics Education* in the last 6 years were related to modelling"(Kaiser & Stillman, 2018, pp. 9-10); in its presence in international tests like the *Program for International Student Assessment (PISA) 2012* (Organisation for Economic Co-operation and Development [OECD], 2014), and for example in its inclusion in the curriculums of countries such as Chile (2009).Therefore, a broad range of research has branched out into different topics (see the next section).

8.1.1 Mathematical modelling as a cycle[7]

Mathematical modelling may also be a learning approach that uses elements of real-life situations to create models with mathematics. In this approach, students work together in a cyclical process that involves different stages (see figure 15), such as formulating a mathematics problem based on a real-life situation, setting up a mathematical model that explains the problem, attempting to find a mathematical solution for the problem, explaining the model, interpreting the solution and comparing the solution with the original real-life problem (Mason & Davis, 1991; Blum *et al.,* 2009; Lawson & Marion, 2008).

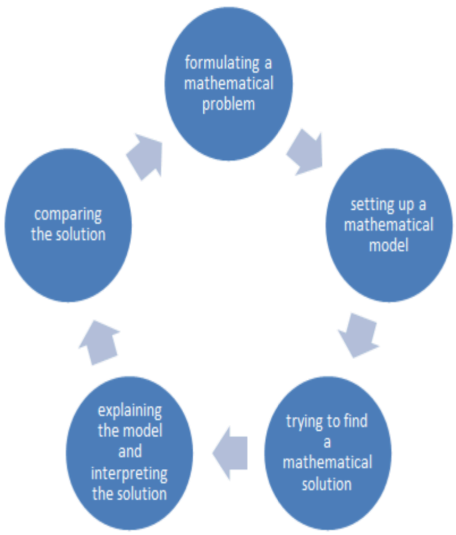

Figure 15: The cyclical process of mathematical modelling (based on Mason et al., 1991; *Blum et al.*, 2009*; and Lawson et al*., 2008).

[7] Part of this literature review is from my paper, Teachers' beliefs about mathematical modelling: An exploratory study. In T. Dooley, & G. Gueudet, (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education.* (See appendix 1, pp. 270-271).

8.1.2 Mathematical modelling as a competence

Maaβ (2006) stated that, to carry out a mathematical modelling process, it is necessary to have the following abilities and skills: to structure real-world problems; to work with a sense of direction towards a solution; to argue in relation to the modelling process; to write down this argumentation; and to see the possibilities mathematics offers for the solution of real-world problems in a positive way. In addition, those competencies will be part of the day-to-day activities in the lesson.

Other studies, based on constructivism, suggest that it is necessary to pay attention to the pupils' mathematical thinking styles because "mathematical thinking styles influence the way in which the modelling process is carried out" (Borromeo, 2007, p. 268). For this reason, pupils who are "visual thinkers" can work in the face of real situations, move to a situation model and then return to the real situation.

8.1.3 Teaching mathematical modelling

Kaiser (2006) suggested that, while mathematics teachers were willing to consider modelling for daily school activities, actual practice showed that this topic played only a minor role. Other studies suggest that "teachers who are novices or have not enough experience in modelling can have difficulties in this process" (Tekin Dede *et al.*, 2016, p. 1). In addition, concerning teachers' beliefs about mathematical modelling (Ramirez, 2017), teachers did not feel well prepared; however, their beliefs could also be linked with some aspects of mathematical modelling, such as a problem being taken from a real situation.

In the paper, *Mathematical modelling: Can it be taught and learnt?*, Blum *et al.* (2009) suggested ways in which teachers should implement the mathematical modelling cycle. Mousoulides (2009) suggested teaching mathematical modelling with an emphasis on different aspects, such as selecting an appropriate task and teaching methods; developing modelling competencies; and using information and communication technology (ICT). Integrating technology in a modelling task can provide “a rich and motivating environment for learning mathematics” (Ghosh, 2016, p. 4). In addition, Tekin Dede *et al*. (2016) suggested a guide for a modelling application between teachers and researchers based on planning, implementation and assessment.

However, others believe “modelling is something you learn about or “catch” only by doing” and that only “you” can do the doing (Manson *et al*., 1991, p. 51). Therefore, it seems to be important to support the modelling process cycle, and this support, from my enactivist point of view, will depend upon each teacher’s approach to mathematical modelling.

From this stand point, our autonomy (see chapter 3, section 3.5, pp. 30-31) and our unique way to act emerges in relation with the environment. The modelling cycle is a framework, a way to categorise behaviours, such as, applying and comparing. There is, therefore, no one way that the modelling cycle can be used, like a computer machine adding or sending information.

The way that maybe each teacher and their students maybe act when they are working on a mathematical modelling task is firmly linked to their usual experience in the mathematics classroom, their historicity and the fluctuations and perturbations of events in that moment (see chapter 5,section 5.2, pp. 67-71).

8.1.4 Mathematical modelling task

A mathematical modelling task is a task that involves a mathematical problem, with questions that have been adapted, usually, from a situation from daily life, for example profit and taxes. The aim of a modelling task is often linked to working within the mathematical modelling cycle, as described in the previous section. In my opinion, we can also work with a mathematical modelling task without mentioning the cycle, because, for me, mathematical modelling is a mathematical task.

There are a variety of modelling tasks for age groups and different mathematics, including in some cases the use of digital tools to support on the task. Examples of this could be seen in the recent ICME (2016) conference, i.e., Two drug examples, using STELLA[8] software (Fisher, 2016, p. 2); and using game design to engage in the modelling of a real-life changing ecosystem (Caron & Lovric, 2016, p. 1).

So, why work with the teacher and their students on a mathematical modelling task within this study? A mathematical modelling task offers the opportunity to solve a mathematical problem in an 'open way'. The task is composed of many mathematical

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[8] STELLA, meaning *Systems Thinking for Education and Research*, "is a modelling software package that diagrams, charts [sic], and uses animation". Online at https://www.dataone.org/software-tools/stella-systems-thinking-education-and-research, accessed 20th June, 2018.

variables and, perhaps, once the problem is solved, the students and their teacher might find that their answers are not necessarily the same.

This uncertainty in mathematics, of differing methods and solutions, provides a platform to observe how mathematics learning emerges in the interactions that teacher and students have when they are exposed of this prompt.

8.2 Making decisions about the site of the research

8.2.1 Choosing a school using the Chilean mathematics curriculum

Between 1990 and 1998, the foundations of the national curriculum in Chile were set, and these are still in use. However, in 2009, the curriculum was updated to meet the requirements of an evolving society. This led to a profound reform of the institutional framework of education, creating a new law (Education General Law, number 20,422) and therefore a new national curriculum. Currently, Chile's curriculum is in a transition period between the Curriculum Framework and the new national curriculum called the *Curricular Basis* (Ministerio de Educación de Chile, 2012a; 2012b; 2013; 2015) the implementation of this new curriculum has been undergoing a period of development.

In 2012, the process of implementation started gradually, from year 1 to year 3. Then, in 2013, it was carried out from years 4 to 6. In 2016, it was implemented in years 7 to 8. The process for year 9 started in 2017 and that for year 10 in 2018 (see appendix 8, p. 282).

It is important to note that the term implementation refers to the current law establishing the Chilean national curriculum. Chilean schools are teaching a set of aims such as: mathematical skills (i.e., solving problems, representing, modelling, communicating and making argument); thematic lines (i.e., number, algebra and functions, geometry and probability and statistics) and aptitudes (i.e., working collaboratively, being critical and curious, resilience, being responsible in the use of the technology). In thematic lines the syllabus, mathematical modelling is also specified as a type of mathematics. (Ministerio de Educación de Chile, 2016, p. 97; p. 101, translated from Spanish).

In both implementations (primary and secondary school), one of the issues is that this new curriculum promotes mathematical modelling as a skill and content to be incorporated within the school. In Chile, the use of mathematical modelling within the lessons is quite novel. Although the specification of mathematical modelling is new, teachers and students have been doing some aspects as part the old curriculum, such as using formulas describing real life to solve problems.

My intention is to look for the emergence of mathematics learning in the pathway of interactions between a teacher and their students (specifically in conversations). This will include when they are working with a mathematical modelling task. I want to see the current situation in terms of the implementation of the national curriculum in Chile. I have decided to work with a school in which I can see evidence of good outcomes in the national test for mathematics.[9]Thus, the selected

[9] Since 1988, the national assessment measures learning outcomes according to the national curriculum (currently the subjects involved are Communication and Language; Mathematics; Sciences; Geography, History and Social Sciences; and English Language).

school will have shown evidence of developing mathematics in accordance with the national curriculum. In addition, as a researcher, accessibility and familiarity with this type of school helps me to be more involved in the process of observing/researching because, before starting my doctoral research, I was working in one of them.

8.2.2 Choosing a mathematics classroom in a Chilean school

The Chilean mathematics curriculum seeks to promote the development of forms of thought and action that enable students to process information from reality so they can understand this information and the concepts they have learned more deeply (Ministerio de Educación de Chile, 2012c, p. 87, translated from Spanish). Clearly, this goal from the national curriculum can be related with a mathematical modelling task in terms of taking a problem from reality, and then explaining and applying this through the mathematical model. In addition, in both the primary (years 1–6) and secondary (years 7–10) mathematical curricula, modelling is highlighted as a skill to be developed.

On the one hand, in the curriculum for the first years, the students build, simplify and abstract a version of a system, usually more complex but catching the key concepts and explaining them through mathematical language. So, from mathematical modelling, the students learn to use a variety of data representations and selections, as well as appropriate ways to apply mathematics methods (Ministerio de Educación de Chile, 2012c, p. 89, translated from Spanish).

On the other hand, in the curriculum starting from year 7, the goals include "students learning to use a variety of ways of selecting and representing data, selecting, applying mathematics methods properly, and using adequate resources to solving problems to make sense of equations, functions and geometry” (Ministerio de Educación de Chile, 2016, p. 98, translated from Spanish). Therefore, there is a clear intention in the Chilean curriculum to work with mathematical modelling, and this has been developing gradually since 2012. This means that in 2017, students who started with the new curriculum in that year will be between sixth year (11 years old) and eighth year (13 years old), having completed five years of academic studies under the current mathematics curriculum.

Therefore, to address my research question in the Chilean context, I considered the stage of transition of primary and high schools (see appendix 8, p. 282). For this reason, I have chosen to work with students in the eighth year (13 – 14 years old), who will be studying with the current curriculum.

Working with students at this level offered the opportunity to generate more in-depth research with respect to how the students will be adapting to the situation in mathematics and what sense they will make when they are doing mathematics. This is because, each student has a background of at least five years learning the new national curriculum, helping me to contextualise mathematics according to their reality.

This will be the second generation in the school to carry out this curriculum in the eighth grade, which gives me the opportunity to observe how the students work with a mathematical modelling task taking into account the experience that the teacher has lived with this new curriculum. As, a researcher, I have some familiarity with this curriculum, because I was part of the team that worked on its development in the Chilean Ministry of Education and, as a mathematics teacher, I usually worked with students of this age. Therefore, in my view, my familiarity with the context will help me as a researcher to be more involved in what I am doing, understanding more deeply what is happening for me in the process of learning about learning mathematics, when observing the major details of my surroundings. These experiences will also help me to bring my descriptions to others (see my metaobservation, in section 9.7, pp.188-189, about taking account of my actions as a researcher), bringing forth what my coherent actions are through my observations that explain my own actions as an observer who is observing a mathematics teacher and their students (based on Maturana, 1988, p. 3).

At the same time, I recognise that familiarity with the context could produce prejudgments about what 'must be'. However, from my enactivist position, any act of doing is a new act, a world of possibilities in a mathematics classroom for that reason, I will try to keep my observations close to the details of the interactions, such as, the actual exchange of words in their conversations, rather than say: "They are speaking on fractions".

8.3 Working with conversations about mathematics in a Chilean School

Through a mathematical conversation, which is a type of interaction within the classroom, students and their teacher bring their ideas about a mathematical task. In asking questions about the task, making suggestions, and so on, they are showing their unique way to approaching that kind of prompt.

Naturally, conversation is not the only way to interact. We can also interact through making a gesture with our body and perhaps do not accompany this action with speech. All these actions bring forth a world that could arise to us through conversations (see chapter 3, section 3.3, pp. 27-29). For me, one of first characteristics that we acquired as a human being in order to communicate something to the others is conversations, which arises through a chain of coherent behaviour between each person and their surroundings (see chapter 3, section 3.9.4, pp. 47-51: *coherent behaviour).*

I am interested in conversations because conversation is a common medium for interaction in the classroom, as "classroom talk […][is] not just a feature of learning but one of its most essential tools" (Alexander, 2004, p. 19). Conversation within a classroom is a common daily activity in Chilean schools, where this research is carried out. The basic curriculum encourages the teaching and learning of the mathematics curriculum through "skills", such as, "make an argument and communicate", the students using their speech and mathematics symbols to describe, explain, assess and bring an argument within a mathematical situation (Ministerio de Educación de Chile, 2016, p. 112). Although conversations could be part of the daily activity in school, it does not mean that the curriculum across the work encourages the conversation.

8.4 Why a teacher and their students in conversations?

To initiate any interaction, it is necessary to have at least one pair (i.e., object and person, or between two persons). Similarly, with conversations (which are a type of interaction), to start we need at least two persons being involved in it. The conversation can be between a teacher and students or between the students themselves (i.e., Sfard, A.,Nesher, P., Streefland, L., Cobb, P., & Mason, J. 1998).

I have decided to focus on both cases, teacher and students and between the students themselves, because I want to explore the emergence of mathematics learning in rich detail from different perspectives from the participants within the mathematics classroom.

8.5 Looking for a school that match with my criteria selection

8.5.1 Criteria selection

In the section above (8.2: Making decisions about the site of the research and 8.1: Mathematical modelling) I have described my reason for working with Chilean students in the eighth year (13 – 14 years old), given that they will be studying with the current curriculum.

In addition, because this is a qualitative exploratory study about my learning of the emergence of learning through interaction in the conversations between teacher and their students and amongst students themselves, based on an enactivist approach, the attention to detail needed in these interactions, contributed to consider working with a small group of class.

In order to go more deeply into my data, rather than a brief description of these, an important part of my research (as I discussed in chapter 6, section 6.1, pp. 76-78; chapter 7, section 7.2, pp. 106-108) is the time taken for the classroom observations and observing the observations. According to Mertens (2010, p. 257), "there is no hard-and-fast rule that says how long a researcher must stay at a site". For that reason, I will spend time with the school according to mediated conversations with the teacher and the school and its study plan. For the time in the school, it will be required that the students and teacher will be studying a mathematical modelling task.

According to my professional experience in the Chilean context and the organisation of the national curriculum of Chile, the academic year in Chile starts in March of each year, named the first term, with a period of two weeks of winter holidays in July. The students return for their second term, with a period of holiday starting in December. In order to maintain a prolonged engagement (Mertens, 2010, p. 256) in my research, I have decided work in the middle of the first term with the students and teacher, because at that point they will have gained some familiarity with the new academic year and before the winter holidays. As Mertens (2010, pp. 256-257) pointed out, "conducting observations in a classroom the week before winter holidays is probably not going to give the researcher an accurate picture of typical student and teacher behaviors [sic]".

8.5.2 Contacting a Chilean school

The empirical procedure used in this research, in order to contact a school in Chile, was electronic mail with a letter attached about my research, inviting the school if they would like to participate in my study. The schools selected to send the email to, had been chosen according to the mathematics content in the national curriculum and that the students had achieved high scores in mathematics according to the national test, SIMCE (Sistema de medición de la calidad de la educación or translate to English measure *system of measurement of quality of education* administered by the Ministry of Education, Chile).

SIMCE records are public results showing, each year, the progress of each school according to the national curriculum proposed by the Ministry.

After spending two months looking for a school, one of them accepted to be part of my research and I was invited to a personal meeting with the headteacher in Chile before collecting the data with the teacher's head before to start collecting the data (see chapter 8, section 8.8.3, pp. 137-139. *Working gradually and iteratively in the process of collecting data*).

8.6 School context

The educational system in Chile has two options once a student is 13 years of age: either attending at lower and upper secondary school education or alternatively, a Technical (vocational) school. The school of this study is described as lower and upper secondary school education and follows the national curriculum established by the Minister of Education of Chile.

The school offers a programme of study from reception (five years old), primary (aged six to 11 years old), lower secondary (12 to 15 years old), and upper secondary (16 to 18 years old). Once these levels have been completed, the student completes a national test with the goal of attending university. The school, of this study, is a well-recognised private faith school in Chile, founded in the 18th century, with high achievement in the national academic test, one of my reasons to choose this school.

8.6.1 Mathematics staff

In conversation with the director of studies, it was explained to me that there are six mathematics teachers (they studied mathematics as a bachelor's degree, including pedagogy as a subject) who teach fifth-grade students (aged 10) and students up to high school. In the first two years (six-to-seven years old), there is no special mathematics teacher; that is, the same teacher teaches all subjects to students. However, from the third to fourth grade, students have a primary teacher who studied mathematics as a specialist subject.

In addition, the school has other mathematics teachers that provide remedial lessons. Students attend these lessons once a week if their score in mathematics is below 5.5 (the Chilean scale ranges from one to seven and the passing point is four).

8.7 Ethics

8.7.1 Concerns and ethics consent

According to Clough and Nutbrown (2012), "Ethical issues are central to the methodology of any social science study" (p. 173); Undoubtedly, there is a moral commitment between the participants and my research, which can be demonstrated by the way I, as researcher, doing with them, through the collecting of data and also through the process of writing about my research in a thesis, article and presentation, in fact, literally in any action that involves myself with the study.

To be more specific, before going to collect data for this doctoral research, an ethics form concerning this process was completed after discussions with my main supervisor, Laurinda Brown, in December 2016 (see appendix 3, pp. 272-275 : ethical discussion) as a part of the requirements from of School of Education (SoE), my current university. In this discussion, we have used the prompts recommended by the SoE such as, researcher access/exit, information given to participants; participants right of withdrawal; and informed consent amongst others (see appendix 3, pp. 274 - 277).

The ethical decisions in actions involved in this research, can be observed, for example, on each transcript, through a configuration that protects the anonymity of the participants (chapter 9, section 9.5, pp. 182-186). Especially, we discussed researcher access taking account that this study was outside of the United Kingdom, reflecting access to participants in accordance with the school rules and the Chilean Law, including parental and teacher consent (see appendix 4, p. 278).

The process of the interviews (chapter 8, section 8.9.1 and section 8.92, pp. 140-146) was carried out within the school context in order to maintain respect for the participants' usual environment. In these interviews and again in order to maintain anonymity no names have been recorded.

In addition, the responses of participants and academic bodies are presented and maintained without bias from my own assumptions or judgments of the researched situation. In order to maintain the anonymity of the participants, respecting their response, the transcriptions of the interviews and audio transcription of the mathematics classroom are not part of the appendix section.

Following ethical procedures, videotape and audio recordings of the interviews are encoded and stored digitally, as promised when receiving consent from the participants.

To ensure that ethical considerations are followed into action, it seems necessary to define the parameters I have been following within my research. I decided to base all actions that involve ethical decisions within my research on the principles of independent-autonomy and trust decisions. In this case, independent-autonomy decisions are defined as “access to research, voluntary consent and procedures for securing consent” (Punch *et al.,* 2014, p. 76). Because I was working with minors, teachers and a school, I have obtained the necessary consent forms and, in the case of minors, a consent from a parent (or his/her guardian) (see appendix 4, p. 278). Trust is defined as “data protection, privacy, legal requirements of the participants and any related documents” (Punch *et al.,* 2014, p. 76). Each participant needs to understand the proposal and research methods planned for my study. For this reason, this information was explained (see chapter 8, section 8.8.3, pp. 137-140: *Working gradually and iteratively in the process of collecting data*) to the participants according to their needs, including the school which I worked collecting data. The institution will receive feedback when the project is complete if they require this information.

“[T]he condition in which participants understand and agree to their participation without any duress” is a fundamental part of conducting research (British Educational Research Association, 2011, p. 6). In this sense, each participant will be able to decide if he/she wants to participate in the study or change his/her decision during the research.

Finally, I sent an ethics checklist application for its reviewing to the ethics committee of the University of Bristol. I received the approval from them in January 2017, before going to collect the data of the current study (see appendix 2, p. 272).

8.7.2 Ethics through action: A meeting that never happened

In section 8.7.1 I explained my approach to ethical concerns, and, briefly mentioned the idea of *ethics through action* in my research. Now, I would like to explain in more detail what I mean by ethics through action.

I once read a book on *Ethical know-how, action, wisdom, and cognition* by Varela (1999). In the text, the concept of experientiality lived in an act, meaning literally acting on what can be seen as 'ethical' in the eyes of the observer, caught my attention, especially the question, How can I recognise when am I acting ethically or not? By this, I mean other than by established 'rules' of behaviour that I can name as ethical (based on the Oxford dictionary, 1995, or any requirement that I need to follow in a research ethics procedure).

Let me explain with the next story, referring to a meeting that never happened.

The school where I collected my data has their own procedure for requesting authorisation from the parents/guardians, for example, for cultural travel outside school. The way this works is that the main tutor from each course sends a printed letter to each of the parents/guardians asking for their authorisation for any activity that involves the students.

When I arrived at the school, they asked me if they could re-write my consent in the form in the format of the school. I accepted this change (my ethical decision) because, on the one hand, I would be able to read it and observe if this new consent form was according to my research and requirements. On the other hand, although I have received ethical approval from the ethical committee of the School of Education in the University of Bristol, I have described my way of working with openness and an iterative process that emerges through interactions with others, in this case through the interactions with the participants from this particular school.

In the process of collecting the forms for the purpose of receiving approval from the parents/guardian and teacher, the school requested that I enter the classroom, where the approval was being sought. However, I preferred to wait for the approval of the 23 students and teacher before I went inside. I mention this story, because this is another decision related to an action, to wait for the consent forms, which shows my ethical commitment to the participants and to myself.

This situation of waiting could be named an ‘ethical action’. It had not been planned ahead; rather, it was an event that occurred through interaction with others, and where my change occurred within my way of acting.

Within this waiting time, I received the approval of 22 of the parents/guardians, the teacher, and the support from the school. However, one of the parents/guardians decided not to participate in my study. For that reason, the school offered to let me attend the classroom, considering I had the consent of the majority of the participants. The school would ask this particular student to work in the library when I would be there.

Clearly, this suggestion was not prearranged. I felt discomfort with the pressure from the school on one of the participants if the school were to ask this student to leave the lesson when I was there. Although it would mean that I could collect my data, at the same time, how could I leave a student outside of their usual classroom, even knowing that I have the support of the school?

I decided again not to go inside of the classroom (my ethical action) until I could have full approval or at least the student (with no approval) did not attend for another reason (i.e., not attending the school for a personal reason). I think, although being inside the classroom was necessary for collection of the data, I had to have a moral balance (based on the respect for one another) between my research and the persons whose behaviour I would be studying.

I believe in building a trusting relationship with the participants, taking a position of a conscious listener in my actions and their actions. For that reason, I suggested that the school invite the parents to come to meet me, to learn more about the project and ask me questions or concerns about my research project and data collection. The school invited the parents/guardians on my behalf, but they sent a note, as you can see below (figure 16) saying that they would like to participate in my research, so in the end, the meeting never happened.

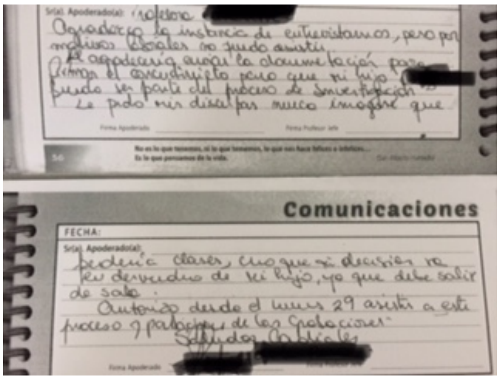


Figure 16: Informed consent through note (in Spanish). Names and signature have been deleted for ethical reasons.

Translation from Spanish to English

Dear Teacher xxx

I appreciate the invitation for our interview however I cannot attend because my work. Thanks from beforehand for sending to me the consent form to put my signature and for my son be part of the research process. I ask you thousands of apologies because I never thought my son could miss lessons. I think my decision [about not being part of the research in the first instance] is going against my son because he must go out from the classroom. I give my consent from Monday 29th to be part of this process and participate in the video-recording.
Best regards
xxxx

Through the history described above, there are moments in which I, as a researcher, needed to make some decisions, for example, inviting parents to a meeting.

Living the tension of my own ethical behaviour means making my own actions interacting with the ethical actions of others, such as what the school said to the parents about the student missing lessons, producing, perhaps, some ‘pressure’ on participants to participate.

I was navigating this ethical dilemma between my actions, and the action of the others, where ‘ethics’ is more than a ‘written-sheet protocol’. I could not avoid action from the school because that is part of their own behaviour and their way to act. However, I was reflecting on my way to act ethically, making choices from a variety of options. (i.e., inviting the parents to a meeting with me to discuss my research project).

There were other ethical actions taken by myself, for example, re-writing the informed consent letter according to the school rules. I accepted the use of written notes from the parents/guardians and not the informed consent sheet because consent writing in notes was part of the way of communicating to parents/guardians from the school. If I use a note as a way to express something to the parents/guardians, the school was alright about it because using this method was their usual manner to communicate (personal between the school and parents/guardians).

From an enactivist point of view, through the decisions I made (my own autonomy) and the change lived through my actions with the world, I am learning as well, learning to act ethically. The unexpected situation described in the story was not planned. There are no ‘guidelines’ for seeing the ‘right’ or ‘wrong’ choices, but the actions speak for themselves in the eyes of others who might be observing and what I was observing as well.

8.8 Data collection

8.8.1 Brief description about data collected

The grade-8 classroom (students aged 13-14 years old) comprised a mathematics teacher with 10 years' experience and his/her 23 students (20 in average of attendance per lesson). The small number of students[10] allowed me to stay close to the details in the interactions between students and their teacher when my observations took place.

The class had the same room for all lessons and the teacher needed to move to teach the lesson in the student room. In total, the students had five hours and 15 minutes of mathematics lessons in a week, distributed over three blocks of 90 minutes, and one lesson lasting 45 minutes.

Within a 2.5-month period, I collected data comprising eight 90-minute mathematics classroom observations, five audio-recording and three video-recordings in which the students and their teacher were working in their usual way to solve word problems, exponents and powers, root squares and percentages. Furthermore, for two of the video-recorded lessons, the class was working on a mathematical modelling task

[10] According to the Organisation for Economic Co-operation and Development (OECD), 2008 the average number of the students' in lower secondary education in Chile is around 30. Online at https://www.oecd-ilibrary.org/docserver/eag_highlights-2010-30-en.pdf?expires=1562147673&id=id&accname=guest&checksum=BB2D1E148FC9BCF6FC9E9CCCDD8956A2, July 3rd, 2019.

that was new to them due to Chile's national curriculum recently having integrated this learning goal (Ministerio de Educacion de Chile, 2012c; 2016).

In addition, in the same period, I interviewed the teacher four times and a group of students, comprised of four and five students respectively, twice. The reason for the choice of students was according to interactions observed within the classroom.

Organisation of desks in the grade-8 classroom

The classroom in which the mathematics lesson took place was a square room, with two main windows. The teacher's desk was in the right-hand corner between the whiteboard and the students' desks. Each students' desk was for an individual, no sharing of the same table (see figure 17). Students were usually sat with a space between them, although, sometimes, some of them preferred to sit together. The room was not mathematically decorated but on a shelf were the currently used mathematics textbooks. In addition, the room was equipped with a data projector.

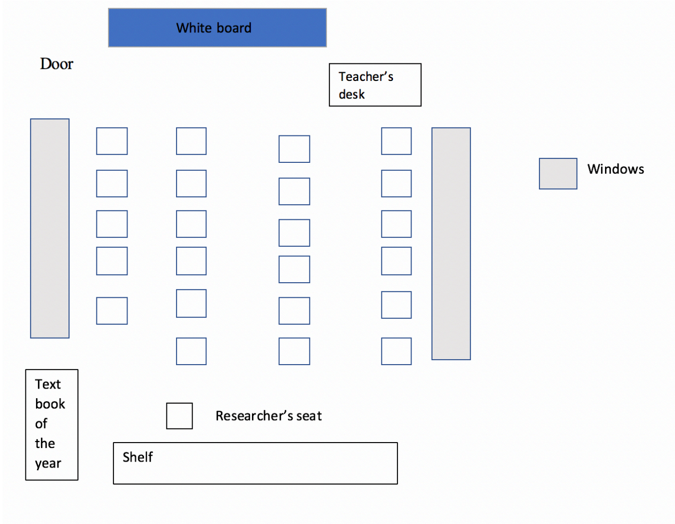


Figure 17: Usual distribution of the classroom.

Sometimes, when the teacher suggested that students work together with others, the usual distribution of the desks in the classroom was changed for that lesson (see figure 18):

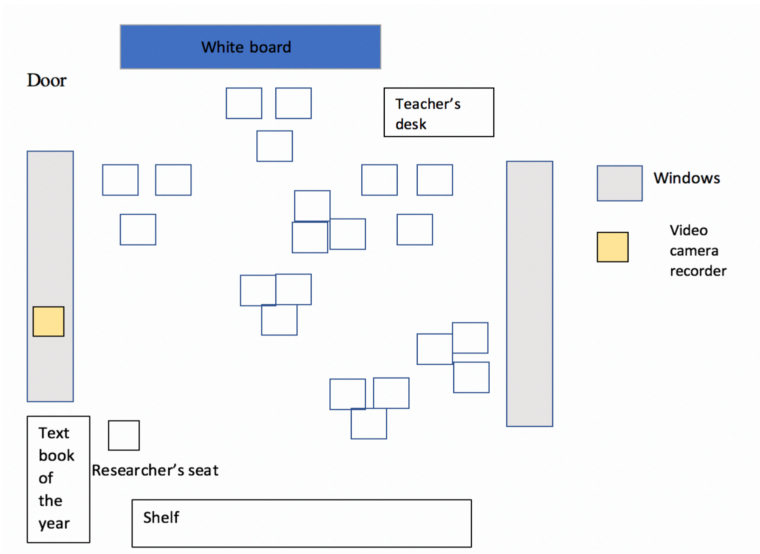


Figure 18: Structure of the room when students worked together with a mathematical modelling task.

8.8.2 Usual classroom behaviour in grade 8

The teacher began lessons by discussing mathematics content from the previous one, for example, reviewing a question (from mathematics exercises), new content, or addressing a word problem. This topic depended on the lesson plan of the teacher.

In addition, students and teacher had an agreement to write the day (i.e., 20/06/2017) with mathematics operations (adding, subtracting, multiplying, dividing) regardless of the mathematics content they were working on at any given time, for example, squaring, and exponentiating.

Over the course of the lesson, the students took notes in their exercise-books and worked at worksheets provided by the teacher or from their mathematics textbook. The lesson generally finished with reviewing of the work that students had done using the whiteboard.

8.8.3 Working gradually and iteratively in the process of collecting data

To Carr (1995, p. 88), "Educational research . . . always involves a commitment to some educational philosophy and hence to the educational values that such a commitment unavoidably entails". This reference induces me to consider my position as a researcher and ask how this can be reflected in my acts as an enactivist researcher; to put it simply, what am I doing in my research and how am I doing it?

Considering the procedures in the collection of the data and inspired by my enactivist position, in which "all doing is knowing" (Maturana & Varela, 1992, p. 26), as a researcher, my position is that I wanted to work in an open environment with the participants of my school, hearing their concerns and being respectful of any protocol from the school, while avoiding any judgments and remaining open to unexpected circumstances.

This way of working in the process of collecting data can be illustrated by the following:

- I had a personal meeting with the headteacher who introduced me to the teacher with whom I would be working in my research. I also presented to the rest of the community of teachers and the head of the school, discussing the proposal of my research and what data I would be collecting, including the instruments for collecting it. In addition, we discussed the parental and teacher consent for being in that classroom, because the school wanted to send a formal letter to the parents, inviting them to participate in my doctoral research project (see chapter 8, section 8.7, pp. 130-133 *on ethics*). I accepted this suggestion, because, as I mentioned previously, I did not want to work in a way with protocols firmly fixed before entering the community. On the contrary, I was aware that, although the school had accepted to be involved in my research, they had their own protocols and rules that I had to respect.
- My first meeting with the teacher took place after the meeting with the headteacher. In this meeting, she showed me what she was doing in the mathematics class, including her annual plan. In addition, she asked me which course I preferred to work with at the eighth-grade level, either the A or B grade

level[11]. I replied, in my efforts, again, to be open to their circumstances, "Please, you choose." She explained her reason for choosing one of the level 8-grade was for the location in the school, with more light coming through the window. Later, as a part of the last interview, she told me: "I like this group [referring to 8-grade], because they always have more questions than the others." I had other meetings with the mathematics teacher, which included discussing the process of unstructured interviews being recorded in the course of each week. We started with a meeting/interview that was recorded, and, later in the same week, I went to observe her class. At the end of an observation day, I discussed with her if it would be possible to have another meeting/interview that would be recorded and which day she preferred. Usually the meeting happened each Tuesday before the main meeting for all the teachers that was part of the school's procedure.

- Access to the students for conducting the unstructured interviews (see chapter 8, section 8.9, pp. 140-146: *Method of data collection*) was also mediated with the teacher. She suggested doing the interviews during a period of her lessons when she would be working on 'old' content, allocating the students to the library. I accepted this suggestion.
- The access to observing the classroom and collecting video- and audio-recordings (see chapter 8, section 8.9.4, pp. 147-148: Methods of data collection) has also been mediated with the teacher. This means that we have discussed when the appropriate time is in accordance with her planning, but at the same time, maintaining my goals as a researcher. For example, observing

[11] In the Chilean context, if the school has two courses in 8th grade, they will be named 8 grade A and 8 grade B.

the students' behaviour when they are doing a mathematics test was not part of my goal, therefore I did not observe that lesson.

Working with this open approach provides the opportunity to build a safe environment of trustworthiness between the participants and me as a researcher. They can be open with their questions and concerns, which also helps me to know and understand their environment in which I start to participate as a researcher in interaction with them.

However, in this type of relationship with an iterative process of collecting data, the dependability of the situation must be established. Dependability of future actions between me as researcher and the teacher or school processes, shows our ways of adapting to the situations in which we are participating.

8.9 Method of data collection

8.9.1 Unstructured interviews

Students and teachers will participate in unstructured interviews, which are "in-depth explorations of interviewees' experiences and interpretations, in their own terms" (Punch *et al.,* 2014, p. 185). This will allow me as a researcher to gain a more in-depth understanding of the meaning of the participants' actions when they are studying mathematics.

It is necessary to privilege the participants' voices above that of the researcher so I will take on the role of an active listener as researcher. Thus, I will keep silent in an appropriate way and pay attention to what others are saying without previous judgment from outside referents or doing something else during the interview. It is important to note that this does not mean I will not have some questions for my participants; however, the questions will be generated through the observation process in my iterative approach to the data collection and my methodological position that learning happens in the interaction and, therefore, it is necessary to consider an iterative rather than a fixed process. Otherwise, how can I note what am I learning as a researcher?

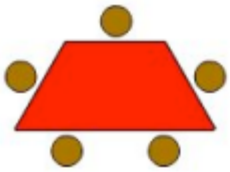
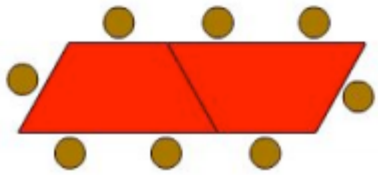
8.9.2 Unstructured interview with a group of students from an eighth-grade class

In the process of data collection and because I have decided to work in an iterative and gradual process, the selection of the students is according to the frequency of interactions observed in the classroom that have occurred between the student and the teacher or other students.

My particular criteria for selecting the participants are: A student who actively participates with the teacher in the context of the whole class, for example asking questions or replying to the teacher; participating actively with other students or in pairs and asking questions or replying to their classmates; other students participating actively with the teacher but in 'private', that is, asking or replying one to one; and, finally, a student who is actively participating with his/her work but does not look involved with the others, the teacher or classmates.

Maheux & Proulx (2015), as a part of their illustration of working on analysis from an enactivist perspective, show how they used a prompt in order to understand how the students solved an equation. They told a group of students how they would solve a particular equation, without solving it with a pen, paper or any computational device.

This method of data collection in mathematics inspired me to start my unstructured student interviews with a mathematical task,[12] which involved determining the relationship between sitting persons and the number of chairs that can be arranged in a long line composed of trapezoid tables as shown below:

> Lipton Elementary School holds an annual tea to honor [sic] the parent volunteers who work in the school. The trapezoid tables they use can seat one person on each of the three short sides and two people on the long side. In other words, one table standing alone seats five people.
>
> 
>
> The tables are arranged in one long row in the cafeteria. When they connect two tables together, here's how the seating looks:
>
> 
>
> 1. How many guests can sit at 5 tables connected in a row?
> 2. How many guests can sit at 20 tables connected in a row?
>
> Explain how you found your answers. Describe any observations or patterns that helped you.

[12] This mathematical task is part of suggestion of a collection of mathematical modelling problems from the book *Guidelines for Assessment & Instruction in Mathematical Modeling Education (GAIMME)* (2016), Consortium for Mathematics and Its Application (COMAP), Society for Industrial and Applied Mathematics (SIAM). Included here with permission (Appendix 6, pp. 278-279).

In addition, the use of a prompt (i.e., photo) within an interview has been recognised as a powerful instrument to initiate or follow-up on a conversation (based on Harper, 2002). The way I have chosen the prompt to use in the interview is reviewing a collection of mathematical modelling problems (GAIMME, 2016). In this book, the authors suggest different mathematical problems according to the ages of the students, for example this trapezoid problem. For me, it is important to consider a problem suitable for the students, given that mathematical modelling is relatively new in the Chilean curriculum.

To be able to quote the mathematical modelling problem referred to above, I contacted the authors of the guidelines asking for their permission, because the Consortium for Mathematics and Its Application (COMAP) and the Society for Industrial and Applied Mathematics (SIAM) require their consent (see appendix 6, pp.278-279).

My intention, through the use of this mathematical task (a prompt from my point of view) is to determine how the participants are interacting with and making mathematical actions according to this prompt, to observe the emergence of their learning. In addition, trying to solve a mathematical task can give the students an opportunity to work on a familiar, common practice in their mathematics classroom one of the common practices is solving a mathematical task.

The use of a prompt, a mathematical modelling task that the students will work on later in their classroom, will also be part of the unstructured interview with the teacher. In the spirit of openness as a researcher (see chapter 8, section 8.8.3, pp. 137-

141), although I will propose a mathematical modelling task, I will ask the teacher if she wants to use this task or would rather use another.

According to the literature reviewed in section 1 of this current chapter, there is not only one cycle that represents the idea of mathematical modelling. In an article named, *The many faces of mathematical modelling*, Perrenet and Zwaneveld (2012) showed various diagrams used by different authors to represent the idea of the mathematical modelling cycle (see appendix 8, p. 282).

In order to promote dialogue with the teacher about the mathematical modelling process, I will use as a prompt sentences that I have chosen from the article by Perrenet and Zwaneveld (2012, pp. 4-6) also related to the various cycles in the literature.

From Geiger (2011):

- Specify problem
- Assumptions
- Formulate
- Solve
- Interpret
- Evaluated Report
- Technology.

From Berry and Davies (2006):

- Real world problem statement
- Formulating a model
- Solving mathematics
- Interpreting solutions
- Evaluating a solution
- Refining the model
- Reporting.

From Kaiser (1995) and Blum (1996):

- Real world model
- Mathematical model
- Mathematical results
- Real situation.

In addition, I added the cycle proposed by Mason and Davis (1991, p. 51).

From Mason and Davis (1991):

- Specify the actual problem
- Set up a model
- Formulate a mathematical problem
- Find a mathematical solution
- Interpret the solution
- Compare with original situation
- Write a report.

There are similarities between the descriptive phrases, used by these different authors, for cycles, which is not surprising for me because the idea behind all the cycles is the same of trying to find a model that represents a problem associated with reality. Where phrases are similar, I will use just one of them in order to make the prompt clearer.

I will be observing a mathematics classroom in which the students and teacher will be working on a mathematical modelling task. Later, and reflecting on what the students have already done on the mathematical task within the classroom, I will consider the observation of Varela that '[I] cannot account for the past with an act that is supposedly happening in the now' (1999, p. 277). However, I can be considering these events through the trajectory each of the participants has lived. I could attempt to evoke some personal experiences, which could entail recognising what matters to each one and what their focus is when they are remembering past interactions in the moment of being interviewed.

I decided to use re-calling this mathematical task as a prompt in order to promote dialogue with the student participants. This decision brings me the opportunity to confront and compare my own experience as an observer in the classroom, attempting to ensure the transparency of my data, hearing the interviewees' voices more than my thoughts, and therefore trying to avoid unsupported judgments.

In addition, the use of re-calling experiences from the lesson, or other mathematical experiences, will also be used in the unstructured interviews with the teacher.

8.9.3 Unstructured observation

Observation provides a tool to observe the process of adapting to a situation, the changes and decisions that the participants engaged in when doing mathematics and interactions between participants in the class (teacher and students). In addition, an observational method helps to gain access to a setting and directly record information about the setting and participant behaviour (Punch & Oancea, 2014, p. 380). Behaviour can show thought in action when the participants are engaged in the lesson, through the conversations that take place. Field notes about the observations were used to describe conversations between students and their teacher or between students themselves.

For me, a mathematical conversation occurs when the topic of discussion is a mathematical concept; for example, the students or teacher can be speaking about the use of percentages in an activity. I do not consider all of the observed conversations to be mathematical, for example, when students and their teacher or students spoke among themselves about when to bring their homework or asking the time of the next examination.

I chose the method of unstructured observation because, from my point of view, doing so enabled me to observe the interactions and take note of how mathematics started to emerge between the participants, with no need to affix what I observed to a previously determined schedule, as would be the case for structured observation. I am not looking for a specific action, such as, for example: How many times did they raise their hand when they were solving a mathematical problem? Rather, my approach to observing and tracing the history of interactions amongst students was broad in nature, but this does not mean ineffective, because this allowed me to be open to any interactions that happened. I could be present in that moment, avoiding judgment of the situation in which the students and their teacher lived, for example, looking for an external mathematical representation or saying there is a problem with the use of number in this task.

8.9.4 Using video-recorded lessons and audio-recorded lessons

According to Stigler, Gonzales, Kawanaka, Knoll, & Serrano (1999), on a report from a Trends in International Mathematics and Science Study (TIMSS), "video data are vast and will continue to provide rich analysis opportunities for researchers" (p. v). In this sense, the use of video data will contribute to supporting some of my observations (because I will not be able to record all the lessons considering my iterative open process of collecting data mediated by the requirements of the school, particularly from the teacher or parents) in the interaction between the teacher and their students, and the students themselves especially re-observations.

In addition, the use of video-recording is in response to my intention to attend to the details of each observed interaction based on my enactivist approach. One of the advantages offered by this method is the chance to work under a different perspective in my observations, seeing and re-seeing what happens in the mathematics classroom in order to gain a more detailed account of what has been observed.

However, as I must recognise, one of the limitations of my observational approach is that I will not be working with other researchers in the same classroom in order to gather other perspectives on what has been observed. Therefore, working with video- and audio-recordings allows an individual to act as observer for a second, and possibly more, times, to better apprehend what has occurred in the interactions of the participants when their learning of mathematics emerges (for more discussion about this point, see chapter 10, section 10.2, pp. 192-198).

8.9.5 Schedule of fieldwork and data collected

The next table (see table 2) shows a description of the fieldwork realised within this doctoral project. I have organised the information according to date, method of data collection, general content, place and number of students. In addition, I have added 3 rows (16, 18 and 20) related to other activities as a researcher (which are not part of the data collected) but happened within this period.

The benefit to presenting these seminars was in terms of my own learning as a researcher. I was disseminating knowledge according to the literature review and my experience in the fieldwork. I was also trying to explain what mathematical modelling was for me, or enactivism as a methodology. I was also receiving feedback from different audiences (mathematics teachers, postgraduate students) allowing me to be more attentive to their proposals than my own view about my project.

	Date	Method	General Content	Place	Number of students (if this is applicable)
1	26.04.2017	Meeting teachers' head and mathematics teacher, fieldwork note	Knowing the school and staff	Office teachers' head.	
2	02.05.2017	Meeting mathematics teacher, fieldwork note	Description 8th grade course.	Teacher's room	
3	08.05.2017	Ethics concern	Student Ethics consent received	Teacher's room	
4	9.05.2017	Teacher's Interview 1 recorded, fieldwork note (20 min)	Lesson planning in mathematics	Teacher's room	
5	12.05.2017	Observation 1, fieldwork note	Fraction, decimal, recurring decimal	Classroom	17
6	15-19.05.2017	Mathematics teacher health leave			
7	24.05.2017	Observation 2, fieldwork note	Solving problem with decimals numbers	Classroom	18
8	26.05.2017	Observation 3, fieldwork note	Reviewing Solving problem with decimal numbers	Classroom	22
9	30.05.2017	Teacher's Interview 2 recorded, note fieldwork (47 min)	Percentage activity/Planning Mathematical Modelling	Teacher's room	
10	05.06.2017	Observation 4, fieldwork note, audio, video recorded	Solving a bacterial problem with power and exponent	Classroom	22
11	06.06.2017	Teacher's Interview 3 recorded, fieldwork note (48 min)	Solving a Mathematical Modelling Problem	Teacher's room	
12	08.06.2017	Students' group interview (34 min)	Solving a mathematical Modelling problem about seats distribution and persons sitting	Library	4

13	09.06.2017	Observation 5, fieldwork note, audio, video recorded	Solving a mathematical modelling problem about budgeting	Classroom	19
14	12.06.2017	Observation 6, fieldwork note, audio, video recorded	Solving and reviewing mathematical modelling problem about budgeting	Classroom	22
15	15.06.2017	Students' group interview, fieldwork note. (47 min)	Re calling experience of solving the mathematical modelling problem about budgeting	Library	5
16	19.06.2017	I gave a seminar about my research project and Enactivism as a Methodology to Postgraduate Students in Pontifical Catholic University of Valparaiso, Chile.			
17	20.06.2017	Teacher's Interview 4 recorded, fieldwork note (52 min)	Re-calling experience with mathematics	Teacher's room	
18	22.06.2017	I gave a seminar about mathematical modelling a literature review to Postgraduate Students in Los Andes University, Chile.			
19	23.06.2017	Observation 7, fieldwork note, audio recorded	Power and Square roots	Classroom	23
20	27.06.2017	I gave a Mathematics Teachers' Seminar in the school, titled: Curriculum and mathematical modelling in Chile.			
21	30.06.2017	Observation 8, fieldwork note, audio recorded	Introduction Percentage	Classroom	19

Table 2: Schedule of fieldwork

8.10 Metaobservation

All this information, about the iterative process, different perspectives, ethics in action, the teacher and their students, the criteria for selection of the school, and so on, are guidelines for the replicability of the study. What is the importance of making an account of the decision-making related to what I have done in this methodological research? From my point of view, there is a context where the interactions that I am studying took place, responding to the proposal of my research. Knowing this context brings some sense of time, schedule and type of mathematics content that the students and teacher were doing.

Why Chilean students of grade eight? As a Chilean and a mathematics teacher I was familiar with these two scenarios. There is a personal motivation. I wanted to understand the emergence of mathematics learning. Starting with this context brings an opportunity of going deeper in my own understanding.

I recognise my closeness to this context, there is no anticipation of what will happen because my interactions as a researcher may or may not trigger other actions in my participants. I am part of the environment in the moment of collecting data, for example, observing the mathematics classroom, asking questions to my participants in the interviews in order to find out 'something' about the emergence of mathematics learning.

Finally, I recognise the importance of considering actions in context and methods not as something fixed but as open to new possibilities in the moment of interacting with the participants of the research. This moment also implies learning through the interaction, as in the ethical story mentioned or the iterative way to work with the teacher and the staff of the school.

CHAPTER 9: My historicity

9.0 Introduction

In chapter 3, section 3, I mentioned that uniqueness of knowing is part of the historicity of each one of us in interaction with the world, quoting Varela (1999) and Depraz, Varela & Vermersch (2003).

In this chapter, I explain my own pathway and historicity once the data has been collected. The decisions I have taken are described through a focus on each of five actions; reading the data collected from my fieldwork notes; starting to transcribe the unstructured field notes, choosing some of the many stories to tell; transcribing only conversations between the participants taken from my field notes and starting to categorise them; defining mathematical episodes and what I will be noting through a conversation between the participants; and linking mathematics episodes with fragments of the interviews. This is the process of selecting data, looking for major details that I see in the interactions and conversations between the teacher and the students in a mathematics classroom.

Explaining my own pathway of interactions shows how I started to make the shift in my action and therefore my learning in this process. As I said in chapter 7, enactivism as a methodology is a theory about learning about learning (Reid, 1996, p. 205).

At the end of this chapter, there is a metaobservation about the iterative process that I have gone through with the data collected from unstructured interviews with the teacher and with a group of students; observations; and video- and audio-recordings of the mathematics lessons. Considering my interaction as a researcher to be as a form of learning in this ongoing process.

9.1 Brief description of the process of selecting the data

I described the observed school in chapter 8, section 8.6 (p.126) and section 8.6.1 (p.127), and I have defined my approach to collecting data in section 7.4 (p. 109); however, what can be seen from the interactions between the participants I observed? As a researcher, I must realise that any decisions about this made through my observations in the classroom and later with the data collected are shaped by my position as observer.

I attended a seminar given by David Pimm entitled, *Speaking mathematically*, in which one of the points discussed was the difference between the data and what a researcher brings from the data. How, as a researcher, can I account for what I have done?

As an observer in the classroom, my observations began at the start of the lesson, when the mathematics teacher and I greeted the students. I then retreated to my usual physical position at the back of the room, where I remained seated until the students began working on their worksheets or were using their text- or note-books. At this stage, I moved around the classroom to observe what the teacher and students were doing and take notes about their mathematical conversations.

Later, with the information I received, I had to recognise that my observations were limited by my being human and by the interactions I was having with the environment, which were determined by my structure. Therefore, the changes in my behaviour are determined by who I am and by the environment in which I am living.

According to Reid and Mgombelo (2015), "the organism as a whole is its continually changing structure which determines its own actions on itself and its world" (p. 173). Dependent upon my own actions and on those of the people I was observing, I was part of that environment but with a different role, an observer. From an enactivist perspective, this is known as *structural coupling* (see chapter 3, section 3.9.2, pp. 45-46). I was part of the environment, and the environment was part of me. I can establish, as an observer within the environment, in this study the classroom setting, that my constant interactions with the world could demonstrate coherent behaviour (observing a chain of actions that are interconnected). However, at the same time, behavioural coherence is also shown in the interactions (based on Maturana, 2000) between students and the teacher. As the researcher, I am living this loop between what I am observing through my own coherence and how I am observing the coherent behaviours of the participants in the study.

What can I see (or bring through) from the observed interactions, particularly the students' conversations and the conversations between the students and the teacher? From my enactivist perspective, I can see patterns of behaviour that are linked with the participants' coherent actions when working in the mathematics domain.

Considering my enactivist approach, I have chosen some episodes from my participants' interactions. I consider the historicity in their interactions, paying attention to the details and the coherence of the behaviour (mentioned in chapter 3, section 3.9.4, pp. 47-51), particularly during episodes where questions are used in mathematical discussions, such as when a mathematics question[13] arises from the teacher or the student or when they are solving a mathematics problem.

In this context, changes can be observed from different perspectives depending on the focus of the observer, and these can be noted through the "coordination of the knower and the environment" (Proulx & Simmt, 2013, p. 66), as discussed previously in chapter 6, section 6.1, pp. 76-78.

To analyse these questioning episodes, I have been working from different angles. In the first instance, I observed without looking at something in particular. I wanted to maintain my openness to the data and to avoid any judgments in relation to the interactions collected (see analysis section, chapter 9, section 9.7, pp. 188-189).

I then transcribed what I had observed from my fieldwork notes and compared the findings with the audio- and video-records. As I mentioned previously (chapter 9, pp. 152-189), while I spent 2.5 months observing in the classroom, not all of the observations were recorded due to the agreement with the school (see chapter 8, section 8.83, pp. 137-139) and the proposal of using this method. While I immediately translated the interactions from Spanish to English, a professional translator reviewed these translations at a later date.

[13] A question is defined as a spoken prompt that provokes a spoken answer.

When first exploring my data, I started to read my fieldwork notes and transcripts of audio- and video-recordings, seeing what caught my attention. Later, I began to categorise the text according to the characteristics I observed, for example, *Instruction*. After that, I transcribed the interviews, highlighting those answers that could be linked with the interactions that I noticed when second-observing the data. This process enabled me to recall what I had observed when collecting the data and to begin making distinctions through the descriptions collected.

In my third approach to analysing the transcripts, I searched for more details. Based on my enactivist position, I used the details to differentiate between the interactions, enabling me to observe the patterns that emerged through these interactions. This required a re-observation of the transcript based on the basic-level of categorisation mentioned by Rosch (chapter 6, section 6.1.2, pp. 80-84).

I started to look for these kinds of questioning episodes, which allowed me to note the emerging behaviour patterns through observing the interactions in the discussions. These behaviour patterns were different; however, they started to be repetitive at different moments in lessons.

Then, in an effort to delve deeper into the details of the interactions observed, I returned to my interview transcripts and considered how the historicity of the persons in the episodes could be linked in terms of what had been said.

In the next section, I will describe what my decisions have been in relation to my collected data (for more specific information about the data, see chapter 8, section

8.9.5, p. 148-150). I will describe how my own learning started to evolve at this stage of my research through the changes in my actions.

9.2 First action: Reading the data collected from my fieldwork notes

Shaped by the enactivist position, one of the aspects that caught my attention was the observation of a phenomenon that is happening but which, at the same time, I am living through, and therefore learning about through this process. One focus of the observations that I am living in this study is concern for the phenomenon of observing the emergence of learning through the interactions in a mathematics conversation.

To begin my observation of the data, my first decision was to read my field notes, which are full of descriptions of conversations that happened in the classroom that I observed. The reason for starting with my field notes is because this was my first approach to recording my data. Although I recognise, I have access to other resources, such as the video- and audio-recorded interviews with the teacher and students, I prefer to begin with what I have learnt from the first step.

As described in chapter 8, section 8.9.5, pp. 148-150, when I began collecting the data in the eighth-grade school classroom, my first action was to take notes. Now, in the process of organising and making selections from the data, my first action was again to read those notes. This action can be linked with my own patterns of behaviour, that is to say, what I have learned from the beginning of my interactions comes from taking notes about the students and teacher conversations about some mathematical topics related to a mathematical task.

I want to concentrate on all the observation sessions that have been mediated through daily conversations with the mathematics teacher who is involved in this eighth-grade course. In my intention to maintain openness in the circumstance of the school and the teacher, I always asked her, which day she would prefer my attendance at the classroom.

This openness with her situation helped me to work without any unwelcome or unwanted forced interventions. For example, one of the reasons that I could not attend some observations was because the students had an extra activity related to the school's anniversary, a religious ceremony or it was the final lesson before a mathematical test.

Given that it is my intention to learn about the emergence of learning, this first approach of reading my notes caused me to relive in a certain way what I had lived when I went to school and to re-capture the interactions on which I took notes. However, reading my field notes again was not enough to go into the observation in more detail, because I started to focus my attention on the mathematical concepts more than the interactions, which naturally brought me to a sense of memory of what I lived at that time and what I learned in that moment. For example, what different approaches has each student taken to accomplish his or her mathematical task, and which of the many questions that happened in the classroom that I observed between the teacher and the students or vice versa do I remember?

9.3 Second action: Starting to transcribe the unstructured field notes, choosing some of the many stories to tell, but where is the interaction?

After that, my second decision linked with the idea of observing and my intention to gain more detailed access to the data that I had already collected. Accordingly, I started to organise my field notes into episodes given, an extraction of which can be seen on table 3 (see pp. 160-166).

With the intention of gaining more familiarity with my data, I again read some organisations of my fieldnotes. While doing so, I noted that in my organisation I was telling the story of what I observed, adopting a narrative style and bringing some context to what had happened according to my view, in which I was telling the story in the first person, as can be evidenced by the next extracted from my notes:

> In the lesson, I observed students were working on converting a decimal into a fraction. I usually heard the teacher saying, "read the decimal first and then write into a fraction".

I chose a narrative style because I wanted to show what I was living as an observer, with a strong desire to illustrate in what way I was part of the environment when the interaction took place.

Sometimes, within this transcript I described the interactions between the teachers and their students, or between the students, as can be evidenced in field note observation 10 (FNO,10), in particular with the next text:

> And she asked how the student is following the numbers, how much bacteria will there be in 24 hours? S20 said, "3×2^{20}", and the teacher replied, "Why?" The student answered, "I related to two." The teacher

> reviewed her calculations on the whiteboard, saying, “In half an hour we have three by two to the power one, then one hour,” and the student said, “Wait a second, I have not written two hours, I need to change this.” Then the student, S20, said, “It’s 3×2^{48} but I do not have the answer.” The teacher explained, “S20. Relate the base two with the number here” [reference to the hour written on the whiteboard]. (FNO, 10)

However, bearing in mind that my interest is in exploring the emergence of learning, by using a narrative style, the organisation of my data was limiting my actions of observing that moment of interaction between the participants. Specifically, I was concentrating more on myself as a researcher, explaining the story, than on the data I had collected and, thereby, limiting my observations. This was especially the case if I started to make comparisons with the observations made in the classroom, which were also video– and audio recordings. From this one insight, I was able to observe more details, for example, in particular, considering the interactions in a conversation between the students or with the teacher.

Considering my limitations observing the data collected by this type of narrative approach, I decided to move on to considering how the organisation evidenced interaction between the participants. Therefore, I removed what she or he had said written in the narrative style to leave only what the teacher and student actually said, revealing a conversation that happened between the participants.

1	In the lesson, I observed students were working on converting a decimal into a fraction. I usually heard the teacher saying, “Read the decimal first and then write into a fraction” when the students were working with terminating decimals. When they moved into working with turning recurring decimals to fractions, the teacher asked about the “rule” to write into a fraction. One student replied:

	S5: I remembered something that is with zero, for example in ... you write 26 above and then minus 6 (referring to an operation on the numerator) T: Why did you decide this number? S5: Because I remembered something with zero
2	When I observed the students working to change recurring decimals into fractions, some of them asked their teacher, "Why 9? Why 90? Always you write with zeros?" Others were saying, "Because it is a law, a rule"; "I don't understand why nine or ninety"; "I do not understand why subtraction". The teacher wrote on the board the questions from students and she wrote $2.6\dot{4}$ into a fraction. Students keep working with the "rule" but some of them were still asking the initial questions. The teacher explains an algebraic procedure for the use of nine in the fraction, asking the students to repeat this procedure. Then, the teacher asked: "What is the title of our lesson today?" and a student answered: "Why nine?", with another saying, "Decimal numbers". The teacher wrote the titled *Converting recurring decimals to fractions* on the whiteboard and defined the "rule" of each one (those which may have a repeating digit, or a repeating set of digits after the decimal point and those which have digits between the repeating digit, or repeating set of digits and a digit or digits after the whole number (i.e., $1.\dot{4} = 1\frac{4}{9}$ and $1.3\dot{4} = 1\frac{34-3}{90} = 1\frac{31}{90}$)These names, are *decimal periódico, decimal semiperiódico* respectively, in the Spanish translation). I heard the teacher say, regarding the definition: "It is not [pure] mathematics language, it is colloquial [language] but it is for your understanding". I observed that the students were still working on changing recurring decimals to fractions. At the end of the lesson, some of the students asked, again, the same question, "Why 9? Why 90? Always you write with zero?" The teacher said that question will be a challenge for them as homework before the next session.

3	I observed that, after discussion on changing recurring decimals into fractions, the teacher introduces a procedure to explain why nine. She said, "I want that you see this" and then she wrote on the board, saying, "I multiplied all [referring to 1.222 ….] by ten. A student said you cannot do it because it is an infinite number". She followed this by showing that the result was 12.222…
4	I heard the teacher say that "a person" invented this, and this "person" thought of subtracting these two expressions, (a) and (b): (a) $n = 1.2222 ...$ (b) $10n = 12.222\ldots$ She added that, with this [subtraction] strategy she made "the recurring number disappear [does not exist]". Finally, the teacher said, "We are going to repeat this procedure". A student James asked Mary "Why did you subtract?". Mary replied, "She wanted to disappear the recurring [number]". I heard one student say, "So, if the number has a digit between the whole number and the recurring digits, I multiply by 100". Two of the students went to the board and explained the same procedure that the teacher had done with similar recurring decimals to those used previously.
5	T: Which is the non-repeating part? S17: Twenty-six T: No [it is] 6 S17: Okay, [then] 6 S4: These are laws [in mathematics] not requiring justification S5: The rule said.

	T: You subtracted what is outside of the dots. "The rule" =
6	I observed that the students were working on operations with decimal numbers and solving problems which involved decimal numbers as well. I heard the teacher stress that, when they divide a number by a decimal, they need to multiply the dividend by 10, 100 and 1000 whatever the divisor. Some students asked others, "Why by 10, 100 or 1000?". I heard the next dialogue between the teacher and a student. S7: Is it ironic here that 13 divided by 0.05 equals to 260? T: How many times does [0.05] go into 13? It [0.05] is [an extremely] small number S7: Ah, [I see it] Then, I observed the teacher explaining an exercise on the board, drawing some rectangles and saying, at the same time, "Here is a case when a whole is distributed (divided) into half parts".  If I divided a whole into half parts  How many parts are there? Two. Then she wrote $1:\frac{1}{2} = 2$; $1:0.5 = 2$. After that, she asked, "If you have thirteen wholes and you are sharing by a very small number, increase or not [amount shared]?" Some student answer, "Yes", others "No". She replied, "The smaller the divisor, the more the answer increases".

7	I observed that the students had been working on operations with decimals, fractions and problem solving. Then, two of them were solving the next equation in the textbook and one of them asked the other, "Why?". The instruction for this problem was: For each case, find the value of x. $$x - \frac{5}{2} = \frac{5}{8}$$ An explanation follows the next dialogue: S2: Look, I multiplied by four there [on the denominator] so I got eight and then, ten minus five is five, so for that reason [the value] is $\frac{10}{8}$ S3: But, that is not right. You multiply by one of them and not both [numerator and denominator]. Look, it is 25/8 [if you multiply numerator and denominator by four] then the value is [because 25 minus 20 is five]
8	I observed that the teacher asked the students to describe different strategies when they solved a doubling bacteria problem (they needed to calculate the amount the bacteria grew per hour until 24 hours). Some of the students say the answer to each question that the teacher posed in the problem (i.e., 24 bacteria after a hour and a half, 48 bacteria after two hours, and so on). Others said, "I added each bacteria considering what happened before", writing her strategy on the whiteboard. Another student went to the whiteboard, saying, "I have done it differently". She explained that she did not want to work with half hours so she started to do first each two hours and then each four hours. The teacher repeated what the student said, saying, "So, she reduced the hour to make less calculations. Have you finished the multiplication for 24 hours?" The student said, "Only until 16 because I did not want do it but it is the same idea". The teacher constantly asked if anybody knows how many bacteria there were after 24 hours. Some students said numbers that were not related to 24 hours. The teacher said that, within this problem, she wanted them to note that it was a tedious process to multiply these

	numbers (without a calculator), some many times and continued, "We, therefore, need a simple expression to say the same, an expression that represents repeated multiplication of the same factor. We have named that power". I heard a student ask, "Teacher, where are you going with this?" She replied, "Good question and I return the question to you, to all of you. What am I looking for?" One student says, "Should be with two to the power of [something]" and another says, "Is doubling by two". Then the teacher said, "What is the title of today's work?" One student says, "A tedious multiplication". The teacher smiled and then she said, "Well, it was tedious and that was my intention". Another student said, "It's powers and exponents".
9	Within the first hour, I observed that students were saying their answers about the reproduction of bacteria after each half an hour, meaning, half an hour, an hour, an hour and a half and so on. S16 went to the board and explained that she wanted to solve each hour, meaning, multiplying the amount of bacteria by four and not by two if she had solved considering half hours. Then, the teacher and students were working to find a power that describes the number of bacteria after 24 hours. The teacher wrote 3×2^{48} at the end. S20, said, "I got $3x2^{48}$, but I don´t have the answer [to know how many bacteria there are]", S7 said, "Teacher, you can simplify the expression in 3×4^{24} to 3×16^{12}". The teacher said, "Yes, which means that you are working in intervals, each six hours, each four hours". S16 said, "I tried to do it on the calculator but I got a large number". S20 said, "It is not a complete number in the calculator later".
10	After some questions related to the amount the bacteria grew in each hour, the teacher wrote on the whiteboard: ½ 3x2 1 ½ 3 x 2 x 2 2 3 x 2x 2 x 2 2 ½ 3 x 2 x 2 x2 x 2 And she asked the students, "How is following the numbers? How much bacteria will there be in 24 hours?" S20 said, "$3x2^{20}$" and the teacher replied, "Why?" The student answered, "I related to two". The teacher reviewed her calculations on the whiteboard, saying, "In half an hour we have three by two to the power one, then one hour and [...] wait a second, I have not written, hour, I need to change this." S20

	then said, "$3x2^{48}$, but I do not have the answer". The teacher then explained, "S20 related base two with the number here [reference to the hour written on the whiteboard]".
11	The students and teacher had been studying powers, exponents, multiplication and numbers within the context of a bacterial problem with the goal to know how many bacteria there will be after 24 hours. I observed that the teacher gave the answer to the students at the end of the work on this example, 3×2^{48}. S1 says, "I would like to ask if that answer is the same as 6^{48}?" The teacher said, "I will answer you with another question and a smaller number" and then she wrote ¿$6^3 = 3 \times 2^3$? S5 said, "It is not the same for PEMDAS (Parentheses, Exponents, Multiplication and Division, Addition and Subtraction)". The teacher said, It is not for that", then she says: "I do not like that is memory of. Please tell me how much are 6^3 and 3×2^3". The student said the answers and the teacher said, "So, it is not the same, we will review properties of exponents tomorrow".

Table 3: Example of organised fieldnotes from the observation

9.4 Third action: Organising only conversations between the participants taken from my field notes and starting to categorise them.

In my efforts to gain familiarity with the interactions and noting my previous concern about my narrative style, I decided to organise only the conversations between the participants and from there, start to conceptualise what I was observing, according to my own coding of some of the observations.

The next table (see table 4, pp. 171-182) was obtained through 36 episodes chosen because of different mathematics contributions from the participants that I observed, showing how I started to work in this 'ideal' of conceptualisation, a way of categorising the levels of action that I started to observe.

So, in regard to my levels of observation in this transcript, I started by interpreting what the action was telling me. For example, for the theme *expectation* (row A), section A1, the teacher said, "[I]t is for your understanding", which, for me, was an expectation from the teacher to their students. In a similar way, in row A, section A2, the teacher said, "I do not like those tricks. You need to know the why of the thing", referring to the use of PEMDAS. By being involved with my data and therefore categorising, I interpreted this action as an expectation from the teacher in which their students do not have to use this type of memorised technique.

I noted that some students were constantly asking the teacher about mathematics, as shown in row B, where I wrote in section B1, "Teacher said . . ." or in section B2, "Teacher can I change the value?" The students were waiting for confirmation from the teacher of what they were doing. I categorise such comments as *dependency*, because I was interpreting that the student was aligned with what the teacher said. There is a sense of acceptance; if the teacher said this, maybe it is correct. Another type of dependency was asking if they can do some mathematics operation.

I categorise row C as *interpretation*, because of the type of words the teacher used in her speech, such as "You relate . . .", in the dialogue between the teacher and a student. In section C2, replying to one student, she said, "I do not have the answer". In section C1, the teacher shares a question asked by Thomas and later refers to his solution. I noted the use of the structure, somebody said 'something'; in this case a student said something, and the teacher picked up on this. The teacher started the dialogue saying, “2^{10} is [equal to] 1024”.

In row D, which is categorised as *instruction* by me, I noted the teacher asking about the instructions of the problem, saying, “But, the question was, “How much is missing?””, after S1 had explained his/her answer (as evidenced in section D1). In a similar way, relating with the instructions in a mathematics problem, D3 shows that the students were looking for a question, such as would be there in the instructions to their usual work when I observed their actions in the classroom.

In addition, in section D2, in contrast to their usual way of working, when the students were working on a problem together and defined some roles, “Who solves it [...], you or me?”, they were not given instructions for this, but they arranged the way to work, “We solved [...] individually and after that, we compared”.

In section E, I labelled *estimation* referring to a numerical estimation, when I observed (heard) that the students and also the teacher were using an estimation technique to find a square root (see section E1 for an example). In addition, I categorised as *verified* when I observed students were comparing their solutions to an equation, as evidenced in E2.

In the label of row F, *undefined root, big and exact number*, I was trying to show in my description the themes on which the teacher and students were working. Following this idea, the label for row G, *memory*, is because student S5 in section G1 said, “I remembered something, that you add zero”. This section gives me a sense of memory, ‘something’ that the students were trying to recall.

I was observing *ambiguity* and for that reason this became the name of row H. For example, in section H1, when the teacher asked their student, "Which integer number is the most similar [to this 0.05]?", the student replied, "Five". The teacher said, "Multiply by 100", having noted this ambiguity between the answer of the teacher and the students.

Sometimes I noted a separation in the discourse that the teacher made to the class. In row I, section I2, when the teacher said, "A person invented this", in that moment I was asking myself, in what sense did the teacher explain that a person invented this? If she is doing the mathematics, independently of what other 'created' this technique, why is she manifesting a sense of not belonging? Is this important in my observation?

I named row J, *triggered*, because, after the intervention of the student, the teacher started to review her own calculations on the whiteboard (a type of triggering that happens through the interaction with the students), saying, "In half an hour, we have 3 × 2 to the power of 1, then in one hour . . . [pause] Hold on a second. I have not written one hour. I need to change this".

Finally, in my account the last row named was, *different approaches to the same problem*. One of the students, S1, wanted to follow the idea of addition, "We need to add all the expenses", while the other, S2, was refuting this idea, "But we do not know how many games we will buy" (row K, section K1).

I was following what the students or teacher were doing in mathematics, but at the same time, given the openness of my holistic approach, I was also situated in different aspects of what I was observing. Through the mathematics instruction about the problem (row D), I was aware of other relationships between students and teacher (rows A, B, C, and I); for example, the specific actions the student performed in a mathematics problem, such as solving an equation or doing estimations, which are characteristics of 'doing' mathematics itself (rows E, F, H, and I).

At the same time, my own history, its meaning and my background as a mathematics teacher and educator began to influence what I was seeing, for example, my way of recognising estimation in finding square roots or verification in solving an equation were intrinsically linked with my background in mathematics. I could not avoid noticing this, but at the same time, I started to question myself: Is this my voice? Or is it their voice? How can I establish the border between my own history as a mathematics teacher and educator and what I was observing in these interactions as a researcher? Although the descriptions that I started to bring to each episode show coherence with the theme title, such as questions (section I1 and I2), others can be labelled in other ways. For example, in section E2, named *verified* because the students were verifying their answers, looking back, I could have made the distinction of *comparing*. By trying to recognise, through the description of what I was observing, what the others have done was leading to me to go far beyond my data. My intention of staying close to details was embarking in a new direction. For this reason, I decided to move to another angle of observation of my transcripts.

As soon I started transcribing the audio- or video-recordings, I started to note the interactions between transcriptions, and the details of what I was looking for in the conversations that they had, which were independent of a theme.

Observing the conversations, a type of action between the participants, led to me reconsidering the behaviours between them. From my current position, although I have sorted out my concern about my lack of proximity to the data collected, I have to wonder, what was I considering within a conversation? What are the types of details that I started to observe in the conversations I have chosen?

Level		Some examples from the organisation of field notes observation and also from the transcript (from audio and video recorded)	
A	Expectation	A1	Regarding the definition of recurring decimals into fractions conversion, the teacher said, "It is not [pure] mathematics language; it is colloquial [language]. But, it is for your understanding".
		A2	T: 3×2^{48}. S1: I would like to ask if that answer is the same as 6^{48}? T: I will answer with another question and a smaller number. $6^3 = 3 \times 2^3$? [teacher wrote on the whiteboard]. S5: It is not the same because of PEMDAS

			(Parentheses, Exponents, Multiplication and Division, Addition and Subtraction). T: This calculation was not for that, I do not like tricks. You need to know the why of the thing. Please, tell me how much 6^3 and 3 x 2^3 is. S5: 24 and 36. T: So, it is not the same.
B	Dependency (if…)	B1	S: Teacher said, so they pay to us to create the game.
		B2	S1: Teacher, can I change the value given [on the budgeting problem]? T: No, you cannot. These are established conditions and you should be adjusted to what is given [on the problem]. S2: Ah! We exceed by 8 dollars. T: You should make some decisions.
C	Interpretation	C1	S2: Teacher, I would like to ask if two to the tenth power plus two to the tenth power [2^{10} +

	(he said, she said, you said)		2^{10}] can be changed to two to the eleventh power [2^{11}]. T: Yes… Nevertheless, $2^{10} + 2^{10}$ would be 2^{11}? [Why would this be the case?] T: Look, pupils, what proposal happened here? S2 asked me if it is possible to change $2^{10} + 2^{10}$ to 2^{11}. I said, "Yes", but actually I want him/her to tell us why he/she believes in doing this change. S2: Because I have [pause], because my [pause], 1024 plus 1024 is [equal to] 2048 and two to the eleventh power is *equal* to 2048. [Silence in the room.] T: Alright. [referring to S2] 2^{10} is [equal to] 1024. He knows [this] and 2^{20} is 1024 and if you added [these two results of the powers] is 2048 and that, is the next base of 2. Yes, he/she could write 2^{11} instead of add two times 2^{10}. Pay attention to what he/she said, added two times 2^{10}. [The teacher writes on the board $2^{10} + 2^{10} = 2$ times 2^{10} and at the same time states the equation, emphasising it in her tone voice.] T: [What] other operation can be calculated […] two times [emphasises through her tone of

			voice *two times*] … two times two to the tenth power? S1: Two to the tenth power by two. T: Two to the tenth power by two. [What] is [the answer to this]? [I can hear some students say a multiplication, [square] root.] T: Hmmm. S1: A power [exponent]. Teacher: Multiply [the] exponent by the same base. [See] what I do? I maintain the [same] bases and add the exponents. This is [a] really good question [S12] [deleted some sentences because they were not part of the context]. Thanks so much. It is true, 2^{10} + 2^{10} is two times 2^{10} [which] is [equal to] 2^{11}.
		C2	S20: 3 x 2^{48} [written on the whiteboard] but I do not have the answer. T: You relate the base two with the number here [referencing the hour written on the whiteboard].

D	Instruction -Usual way of working	D1	T: We're going two persons at the same time [on the board], and they'll say their strategy [to solve the problem]. [After the students wrote their strategies on the board.] S1: Here, I wrote 200 km because it was the finish line. So, if we added all that [the runner] ran 114.5 km, and it is not 200 km, so [the runner] does not reach the goal. T: But, the question was, "How much is missing?" S1: How much is missing?
	-Contrast to usual way of working	D2	S1: Who solves it [referring to the problem], you or me? S2: We solved [the problem] individually and after that, we compare.
		D3	T: Teacher, this problem has no question. S: The budget is the key word.

E	Estimation Verified	E1	"Find the number that, when I multiply that number by itself, I get the square root asked" was the usual resource used by the teacher and students to explain what they had done or to try a new exercise.
		E2	I observed the students working on operations with decimals, fractions and problem solving. Two students were solving the next equation in the textbook, and one of them asked the other why number they should use. The instruction for this problem directed them to find the value of *x* for each case. $$x - \frac{5}{2} = \frac{5}{8}$$ The dialogue below followed: S2: Look, I multiplied by four there [on the denominator]. So, I got 8, and 10 minus 5 equals 5. So, for that reason, [the value] is $\frac{10}{8}$. S3: But, that is not right. You multiply by one of them and not both [numerator and denominator]. Look, it is $\frac{20}{8}$. [If you multiply the numerator and denominator by four], then the value is $\frac{25}{8}$ [because 25 minus 20 equals five].

			*Note: The students and teacher within the lesson had a strong emphasis on decimal numbers, fractions and solving problems. Therefore, solving an equation was not part of the planning for the lesson but it was part of the activities from the textbook.
F	Undefined root, big and exact number	F1	After much discussion, they called for the teacher (T) and asked for the square root of 1,000. The dialogue was as follows: T: The square root of 1,000 is not exact [number]. So, the perfect way to say the square root of 1,000 is the square root of 1,000. S2: So? T: So? What can you make? Maybe, a square root decomposition? S1: I'm still in doubt. After that, the teacher explained the square root decomposition method and wrote the procedure and the final answer [in one of their notebooks]. T: It is less than 10 times 4 and near to 10 times 3 because 10 is just one unit from 9.

			S1: But how much is that number [the square root of ten]? T: The number [of the square root of ten], I don't know, the perfect way to say it is the square root of 10. If you want an approximation, you can enter the square root of 10 into the calculator and multiply that number by 10. S1: [Wrote in his book.]
G	Memory	G1	In the lesson, I observed students were working on converting a decimal into a fraction. I usually heard the teacher saying, "Read the decimal first and then write into a fraction", when the students were working with terminating decimals. When they moved into working on changing recurring decimals to fractions, the teacher asked about the "rule" to write into fraction. One student replied: S5: I remembered something, that you add zero, for example, $2.6\dot{4}$ you write 26 above and then minus 6 [referring to this operation on the numerator[14] $\frac{26-6}{90}$]. T: Why did you decide on this number?

[14] One way to solve this is $2.6\dot{4} = 2\frac{64-6}{90} = 2\frac{58}{90}$.

			S5: Because I remembered something with zero.
H	Ambiguity Way to express what has been done	H1	I heard, “multiply by ten, one hundred, or one thousand”, exchanged between the students and the teacher. The students asked about how to do a division problem when the divisor was a decimal number. A student was trying to divide thirteen by 0.05 and asked the teacher about how to do it: T: Look, it is a decimal number, zero point zero five. Which integer number is the most similar [to this 0.05]? S2: Five. T: [So, we will] multiply by 100.
		H2	T: Show me how you are solving this problem? Which is your strategy? By randomly picking a number from inside of a bag without looking, she chose which students would answer her questions. T: Tell us, please, how you approached the situation?

			Which actions have you used to solve the proposed task? S1: We first spent money on bags and bottles, and that gave us $11.25, and all the difference is for the prize [referring to the game]. And to shoot down the bottle that was higher [than the others], these are three shelves with three prizes: a small, a medium, and a big one. And if [you] shoot down a bottle, will be an amount of money [referring to that each prize will be an amount of money]. We have $150 to spend, and with the bottles and beans [bags], we had minus $11.25. T: Yup, your group adjusted the amount. You're saying that you did not have $150 to spend on prizes so you had allocated $11.25 to spend on… S1: Beans and bottles. T: On the materials, and the difference will be used on the prizes. How will you share that amount on prizes? S1: Because we don't know the amount [of bottles] that [the participants] will throw [from the shelves]. We make all the excess a prize'. T: But, have you bought the prize? S1, S2: No, is money.

I	Questions	I1	I observed as the students continued working on changing recurring decimals into fractions. At the end of the lesson, some of the students again asked, “Why 9?”, “Why 90?” and “Why do you always write with zeros?” The teacher said that those questions would be their homework for the next session.
	Separability of maths and explanation	I2	T: A person invented this, and this person subtracted the expressions (a) and (b): (a) $n = 1.2222 ...$ (b) $10n = 12.222\ldots$ T: With this [subtraction] strategy, she made the recurring number disappear [does not exist]. Repeat the procedure. S2: Why did she subtract? (referring to teacher) S3: She wanted to make the recurring [number] disappear.
J	Triggered	J1	Teacher asked the students to use the examples to determine how much bacteria would be present after 24 hours.

			S20: 3×2^{20} T: Why? S20: I related to two. T: In half an hour, we have 3 x 2 to the power of 1, then at one hour [pause] Hold on a second. I have not written one hour. I need to change this.
K	Different approaches to the same problem	K1	S1: We need to add all the expenses. S2: But we do not know how many games we will buy.

Table 4: Looking for 'ideal' conceptualisation *of the conversation*

9.5 Fourth action: Defining mathematical episodes and what I will be noting through a conversation between the participants

I spent time, in the previous section, discussing such topics as reading my data, organising my field notes using a narrative style and later only organising and transcribing conversations. I had discovered that it is important to mention where the conversation was situated, perhaps using a narrative style. Coding the data based on themes was not representative of the type of interactions that I was looking for, as a way to show in detail what happened between the teacher and students in this mathematics classroom.

Consequently, I decided to focus my observations only on the conversations and avoid starting to code them under a specific theme (my third action). Subsequently, this decision contributed to maintaining my openness when observing my data, (one of my first affordances) and therefore avoiding making judgments of the observed situation.

However, the question then arose: Which type of conversation should I start to note? I decided to work with mathematical episodes that showed to me the participants engaging within a mathematics conversation, such as some discussions or questions that had arisen from the teacher or their students.

In order to note the engagement of the students or teacher in these conversations, I must also consider their historicity[15], the chain of actions carried out, for example, in the conversation between the participants. Once I had identified a chain of actions in the historicity, I could start to note the changes in the conversation for each one and vice versa. This process enabled me to identify the different pathways according to each individual's autonomy and their decision-making (see chapter 3, section 3.3, pp. 27-29 and section 3.5, pp. 30-31).

In addition, hearing the data from the audio- or video-recordings, I started to note the richness of the tone of voice, pauses, faster speaking and emphasis in the voice. This brought to me a sense of the type of conversation in which the students or teacher were engaging. For that reason, I decided to use symbols in the transcripts to maintain the anonymity of the participants and stay close to the data. Following these

[15] The term historicity refers to the knowing that happens in the interactions, observing each participant over longer time frames (based on Varela, 1999; Depraz, Varela, & Vermersch, 2003). This definition is slightly different from history defined as "a written account of past events".

conventions, I configured each transcript from the fragments of conversation that I have named episodes.

The participants in these fragments could be the teacher and their students or just students speaking together. The symbols relate to how I configured each transcript, which can be seen below: Cn: Contribution number within the conversation; /: Represents a second's pause; //: Pause of more than 2 seconds; ~: Faster speaker; Italicised letter: Emphasis in their voices; Sn: Student number; T: Teacher; [ ]: Contribution from the researcher for clarification.

It may happen that S1, S2, and so on will be labelled in different transcripts, but this does not mean that it is the same student all the time.

From the organisation of the interviews, I am taking fragments that represent, to me, what I have observed linking the interactions of the teacher and students in their mathematics classroom. I am following the same pattern to name the teacher (T) and her students Sn. I introduce I, which means the interviewer. Underlined text in the interview, means what I have specifically observed in line with the interactions.

The next transcript is an example of how I will present my data in the analysis section (chapter 10) and what I have defined as an example of micro-historicity[16] (a trajectory observed by me) showing change in the interactions. It shows two students, S1 and S2, solving the following equation $\sqrt{16 \cdot x} = 32$ which involves square roots.

[16] A micro-historicity means to me a set of interactions that can be observed in the transcript in the pathway of each person, through the patterns and decisions made by the person. This definition emerged through my own learning as a researcher in this study, when I was trying to explain what defines a traced pathway in the actions based on my enactivist approach (chapter 10, section 10.3.1, pp. 200-201).

I must acknowledge that within the period of my observation the students had not studied any techniques to solve this type of equation; however, their assignment is to find a number in the equation in order to maintain the number at the end.

C1: (28.00_28.00): S1: Is it the [square] root? [The student has written in his/her notebook next procedure]:

$$\sqrt{16} \cdot \sqrt{x} = 32$$
$$4 \cdot \sqrt{x} = 32$$
$$4 \cdot \sqrt{8} = 32$$

C2: (28.05_28.11): S2: It is. What are you getting for the [square] root [referring to the 8 under the square root] if you know that four times eight is equal to 32? Then this [referring to x]//~, what must it be?

C3: (28.12_28.12): S1: [Deletes [$4 \cdot \sqrt{8} = 32$ in his/her notebook and then writes $4 \cdot \sqrt{64} = 32$]

C4: (28.17_28.19): S2: That's right. // You have done it.

The micro-historicity of the participants begins with a chain of interactions between them. At the same time, as an observer, recognising a prompt (such as, a question, an answer or keeping silence) can be seen as triggering other actions in the conversation in the constant loop between one participant and the other.

In the illustration above are contributions, C1, C2, C3 and C4. However, considering the autonomy of each participant, and therefore their decisions, each pathway is different. Therefore, the historicity of S1 will be according to their actions, C1 and C3, but bearing in mind that the interactions are under other perturbations such as from S2 in C2 and C4.

As I observed the historicity happening for S1, in a similar way I observed the historicity of S2, considering contributions C2 and C4, but bearing in mind the perturbations received by S1, as shown in C1 and C3.

In addition, noting the change in the actions performed by the participants, in this case, S1 and S2, required observing the historicity, but at the same time paying attention to the details of each of their contributions in order to observe if there is a shift in the action or not from one stage to another.

To support the observation of this change and based on Maturana and Varela (1992, p. 40; Maturana, 2000, p. 461), I have identified a *first start* in the interaction, an original state that then changes to a *new start* when they make a shift from the original state (Ramirez, 2017). Thus, in the illustration above, following the trajectory of S1, the first start will be in the interaction in C1 and C2. After this last contribution (C2), especially considering the prompt to S1, "Then this [referring to *x*]//~, what must it be?" The change in the interaction can be noted in C3, when S1 deletes what he has done and the new start can begin.

9.6 Fifth action: Linking mathematics episodes with fragments of the interviews

For the unstructured interviews, I followed the same process as for the observations of the interactions in the classroom. I transcribed the two unstructured interviews with the group of students (each 40 minutes long approximately) and the four unstructured interviews with the mathematics teacher (each 40 minutes long approximately).

I decided to make the transcription myself, not sending to someone else to do. Transcribing myself meant hearing again what the teacher, students and myself said in that moment of the interview. I was approaching the data collected from another view.

After doing the transcription, I read the interviews, highlighting what for me caught my attention. For example, in the next fragment from the interview, the teacher is speaking about how the students were solving the mathematics problem related to modelling:

> I: When the students started their work, what unexpected event did you observe? If you have seen something, maybe, did you think something caught your attention? Something that you were waiting to happen and perhaps did not happen? I don't know.
>
> T: Right, mmm, there is something that [...]
>
> I: or like you named it, the 'click' was triggered [click is a sound that the teacher made supported by her hand rotating close to her head].
>
> T: The insight. I don't know. For me what caught my attention is that it <u>is complicated for them to create [something].</u> You give the tools and then play it that it is not usual for them. Instead <u>they like more a recipe but this is not a surprise to me.</u> But they are in charge quickly. You said this is what we have, create, make a game, advertising etc and in the beginning they say, "What can I do with all this stuff?" But [later] <u>quickly they are in charge [of what to do] and they are still surprised by the possibility to create and to be original</u>, and that I don't like so much.

However, considering my iterative process, I read the interviews again, this time bearing in mind some mathematics episodes that could be linked. I started to build connections between the mathematics episodes and the interviews, building the argument from different perspectives of the data and also exploring more broadly details of the mathematics interactions from the participants of my research (for example, see section 10.3.5.2 p. 229).

9.7 Metaobservation

Based on my methodological approach “learning about learning” (Reid, 1996, see chapter 7, section 7.2, pp. 106-108), I want to reflect on what my actions have been within this research process. The question is: Why do I need to be considering my actions within this methodological study, especially in the selection of the data that I present?

The phrase mentioned in the paragraph above is guiding the impulse of my methodology towards the idea of interaction with the environment. It also supports my criteria for openness in order to avoid creating the representation of a future reality (based on Varela, 1999), not fixing myself in one particular focus of something pre-conceived. Without this idea, how can I observe the loop that provokes the interaction? Later, in the historicity of my interactions, how can I note the loop between my actions and the interpretation of my actions, which I hope will start to evolve as a form of learning in myself as a researcher?

Considering my enactivist approach, learning happens through our interactions with the environment. Paraphrasing “learning about learning” (Reid, 1996), I can refer to “interactions about interactions”. This is necessary because, from an enactivist point of view, any learning happens through the interactions with our environment, including objects, persons and so on. Thus, my learning in this process, the historicity of my own study of learning, emerges through the interactions that I am observing in mathematical conversations between the teacher and their students and amongst the students themselves. Therefore, it is not going too far to also recognise the type of interactions I have made in my historicity; that is to say, what I have done, what has been the loop between me and the environment.

To consider interactions as a form of learning in this ongoing process of the selection of data, I need to recognise what actions I have taken within my research that have led to me making some decisions and not others with the goal of showing how this process has evolved through my actions.

CHAPTER 10: Re-observing conversations about mathematics

10.0 Introduction

In chapter 6 (section 6.1, pp. 76-78), I explained the role of the observer and what has been observed through interactions. I specified how levels of categorisation (based on Rosch, 1978) made by an observer, make explicit what has been observed through the action of observing.

In chapter 7, about my enactivist methodology, in section 7.4 (pp. 109-110) I have described my position on observing and learning as a researcher, in this continuous act of interactions with my study as a researcher. Taking account of the characteristics of my research, re-observing (and therefore learning, see chapter 3, section 3.3, pp. 27-29) the emergence of mathematics learning through conversations in a classroom from an enactivist perspective, in chapter 8, section 8.9 (pp. 140-150), I described the collection of data using unstructured interviews with the teacher and a group of students, and unstructured observations of a mathematics classroom.

In this chapter, I present my approach to working with re-observation of the data collected, now composed into fragmented mathematical episodes of the conversations about mathematics between a teacher and their students or between the students themselves. Through this approach of re-observing, I show how I work with the use of video-recordings as a *second observer*.

I characterised conversations about mathematics when they were emerging through the teacher and their students, and amongst students themselves interacting.

The categories that I am reporting on are: the process of becoming coherently aware amongst students; interval of waiting (suspension moment); each historicity triggers interactions; mathematics empathy through the interactions.

I end the chapter with my metaobservation, noting how tracing the chains of actions of each person allows me to see in more detail how the characterisation of mathematics learning through conversations between a teacher and their students and amongst the students themselves emerged in the eyes of my observations.

10.1 Re-observing

After having spent time on the actions described in my historicity and bearing in mind my intention to provide as much detail as possible, I decided to re-observe my data. The data was composed of fragmentary episodes that showed me discussions around a mathematical task, specifically, questions that had arisen from a teacher or their students and also episodes selected from the interviews that are related with the interactions noted in the fragmentary episodes. Looking for patterns of behaviour, when the interactions took place, allowed me to move from my first actions (in my historicity) to another level of observation.

To re-observe the data, I used one of the categorisation approaches (Rosch, 1978) described in chapter 6 (section 6.1.2, pp. 80-84), which consists of noting an attribute in the object (in my case, the selected writing episodes) and the interactions that I started to have with them. These interactions can be described as my reflecting process, with questioning and comparing between episodes.

10.2 Using videos-recordings as a second observer

10.2.1 A brief description of the use of video in mathematics education research

The use of video as a tool to obtain data in mathematics education research was introduced at least 50 years ago. There is a report, of the investigation of microteaching by Allen and Ryan (1969, cited in Hall & Wright, 2007), focused on the use of video in continuing professional development (CPD).

In the context of CPD, the researcher's use of video to see the teacher's perspective has been highlighted. The processes reported can consist of showing videos to teachers; focusing on the action in the video between the teacher and the students; or/and finding out what a teacher in the research group is observing in the video. The video can be with the teacher's own students or perhaps be a video selected by the researcher. For instance, Coles (2014, 2016) was working with a voluntary group of teachers who showed short videos of their practice in order to enhance the group's discussion about the mathematical actions of the students. Through this practice, Coles explores the role of the facilitator (researcher) and the "skills needed to facilitate discussion" (2016, p. 164) when the teachers are observing the video, finding that "there is an apparent paradox that a move away from judgment is achieved through the use of judgment" (2016, p. 163). This means the facilitator can contribute by giving some prompts to the teachers who are observing the video to move away from their first judgments of what is happening in the details of the video.

van Es, Cashen, Barnhart and Auger (2017) are others who have used video to take note of mathematics instruction from the teacher's perspective while observing, with a pre-designed schedule, during a course on instruction in mathematics that the participants attended. These researchers identified two approaches from the teachers to observing the video of a mathematical classroom: First, "providing detailed descriptions, including responses that consist of a list of observations or play-by-play narrative accounts of what occurred on the video"; and, second, the teachers "note details related to various features of instruction emphasised in the course […] in terms of the framework's ambitious instructions and provided details for each" (p. 173).

In the context of prospective mathematics teachers, a researcher can use video to address the approaches adopted by the teacher to their students. An illustration of this (in the first stage of its study) is the work done by Ulusoy and Çakıroğlu (2018), in which they used two micro-videos (no more than 10 minutes per video) created by selecting the researcher's "mathematical thinking events", in which seventh-grade school students were interviewed by the researcher on the basic "definitions, constructions, and understanding of hierarchical relations related to quadrilaterals" (p. 4) .

The researcher showed these micro-videos to the prospective teachers in order to analyse how they took note of students' mathematical thinking. When the prospective teachers watched the first time, they "only described the events in the video or simply checked the correctness of student's responses" (p. 10). The second time they watched the micro-video, "all prospective teachers provided a mathematical substantial description instead of providing a superficial description of students' mathematical thinking" (p. 10).

The examples of the use of video described above show how the researcher is studying what the others (prospective teachers or teachers) are observing in the video. In this context, of noting what the others are seeing, a study, made by Reid *et al.* (2015), recorded by video the discussion of a teachers' group (participants), which included a researcher as the facilitator of the discussion, when the participants had been observing a video of their own or others' practice related to the mathematics classroom setting. The researcher later "analysed to identify points of difference and similarities observed by the participants" (p. 6) with the use of a second recorded video, noting that "videos play an important role as prompts for reflection on teaching, revealing underlying pedagogies" (p. 14).

In the same context, of using a video as a prompt to the research (Reid *et al.*, 2015) and inspired by the use of video in the classroom setting used in *Trends in International Mathematics and Science Study* (TIMSS) and *Learner's Perspective Study* (LPS) research, Savola (2008) looked at examples of the structures of mathematics lessons in Finland and Iceland. Through the lens of video, she explored the pedagogical function (mathematics relationship between the teacher and their students) and the form of the interaction segments as they happened, that is the students working in groups or on their own with intervention from the teacher.

Using video-recordings of mathematics lessons is a tool for enhancing the professional conversations of teachers. These conversations can be used as research data in mathematics education (Hall & Wright, 2007). In this context of the opportunities to analyse the video-recordings of a mathematics lesson, the use of video can be a tool to encourage teachers to observe and perhaps reflect on their own pedagogical practice (i.e., observing their own mathematics lessons through the use of video) and the practice of others (i.e., observing the mathematics lessons of other teachers). Such observations can support discussion between teachers, facilitating the use of video-recording as a professional development tool for observing the behaviour of students in a mathematics classroom. One common feature in using video-recorded lessons is that all of them required observation from the teachers involved in the research (e.g., Hatch & Grossman, 2009), including the researcher who is observing what the teachers are saying (e.g., Reid, Simmt, Savard, Suurtamm, Manuel, Jun Lin, Quiigley, & Knipping, 2015).

There is a vast literature about using video as a tool where the participants could observe the video-recordings of the lessons and, through that, have access to their "thoughts" about what they are seeing (i.e., Whiting, Symon, Roby, & Chamakiotis, 2018).

According to the description given above, video can be used as an observation tool to enhance discussions between teachers or prospective teachers in research, where the researcher is noting what the participants are saying when they watch the video or as a tool for the researcher to observe what the video has recorded of the teachers' discussions. The proposal for observing the video will depend (of course) on the researcher's goal. However, little evidence has been reported about how a researcher is doing the re-observation of a mathematics lesson in which he/she has participated as an observer.

In the next section (10.2.2), I will explain my position working with the use of video, which I have described as using video as a second observer.

10.2.2 My approach to working with the use of video.

As a part of my methodology and my way of attending to what data I have collected (see chapter 9, pp. 152-188: *my historicity*) in the recursive process of the observations as a researcher, I have used video-recorded lessons (one of my methods of data collection; see chapter 8, section 8.9.4, pp. 147-148) in this study as a prompts to elicit my own reflections and thoughts about what I have observed when I was doing my first observation in the school, and, at the same time, through my observation of video, when I was doing the fourth action in my historicity, *defining mathematical episodes and what I will be noting through a conversation between the participants*.

My approach to working in this way shares some similarities with the description given in section 10.2.1. Specifically, I am using the video as a prompt much in the same way as described previously by such authors as van Es *et al*. (2017), Ulusoy *et al.* (2018) and so on. At the same time, this is closer to the idea that the researcher is observing what the video has recorded, as mentioned implicitly by Coles (2016) and Savola (2008) and more explicitly by Reid *et al.* (2015).

The difference between my approach and the one mentioned in section 10.2.1 is, first, I was an observer researcher when I was in the mathematics classroom, with the lesson being videotaped, participating with the students in a 'passive way' through the use of video, i.e., without leading discussions in the classroom. Later, I again acted as an observer researcher, who is observing what has been recorded through the use of video.

I named this action, “observe again through the use of video what has been observed”, using video as a second observer. I was the same researcher who was in the school observing, and later I was re-observing what I had observed as the second observer. This re-observation enhanced my enactivist methodology about “learning about learning” (Reid, 1996, p. 205).

The changes I started to experience in this re-observation of the video, acting as a second observer, showed my learning in the levels of categorisation that I started to see or recognise from the action of the students and teacher on the recorded video. My focus is not on noting what the participants are ‘seeing’ in the video, or the recorded mathematics classroom. Rather, my focus is on what I am ‘seeing’ in the video, which allows me from the position of what am I seeing to expand my reflections, comparing with my thoughts from my first seeing and also looking for the pathway in the historicity of the interactions of the participants (teacher and students) that I have observed through my other actions, such as re-observing the audio-recording transcripts, field notes and interviews.

This action of acting as a ‘second observer’ of the video (and I place this phrase within quotation marks because I am the same person who is acting temporally in another place observing, but I am still myself) allows me to go into greater detail than when the interaction took place through the conversations between the participants that I observed in the classroom setting. For me, from these details came the richness that I can start to note in the actions performed.

In this approach, I also need to be attentive and open to what is happening in the video-recordings; otherwise, how can I pay attention to (or note) what I am seeing? In this context, I need to educate myself about how having an awareness of the details of the interactions enabled me to avoid making a judgment, “because I have lived this previously, I know what will happen next”.

10.3 Characterising mathematics conversations when they are emerging

Through the process described in chapter 8 and 9, I work in a constant loop with the data, recognising my actions as a researcher, including my decisions around the transcripts selected, for example, reading the data collected from my field work notes; starting to transcribe from the audio- and video-recordings the unstructured field notes; deciding to consider only conversations between the participants; defining mathematical episodes; my re-observation of the data, using for example video as a prompt as ‘second observer’ from my own research (described in section 10.2, in this chapter, pp. 192-198); and the interviews of the participants.

In this context, of observing and reflecting on the types of mathematical conversations that students and teacher carry on in their classroom, I started to note a change in the historicity of the participants. Knowing as coherent behaviour (see chapter 3, section 3.9.4, pp. 47-51) emerges through a chain of actions that provokes learning, when the interaction takes place, allowing the teacher and students to maintain acting in the mathematics classroom observed.

After making this distinction in the actions between the students and the teacher, and supported by my iterative work (see chapter 9, section 9.5, pp. 182-186; section, 9.6, p. 186-188: *My historicity*), through my learning, I was able to start to characterise some of the emergence of learning shown in the mathematical conversations between the teacher and their students, and between the students themselves. To show these characteristics, I have chosen some mathematical episodes, which can be complemented by comments from the participants in the interviews, in order to support what has been observed and how the learning emerges in the interaction.

In addition, I have based my analysis in my enactivist position, specifically using the following concepts (some of them previously introduced in Chapter 6):

- *historicity* (Varela, 1999, p. 7); Depraz, Varela, & Vermersch, 2003, p. 156): considering the trajectory of actions of each participant (students or teacher) (see chapter 3, pp. 27-54)
- *levels of categorisation* (Rosch, 1978) (see chapter 6, pp. 80- 84), and
- *becoming aware* (Depraz, Varela & Vermersch (2000, 2003; Varela, 2000) (see chapter 6, pp. 92-95)
- *empathy* (Depraz and Cosmelli, 2003) (see chapter 6, pp. 100-101)

Finally, because I have taken account of traced pathways in the actions of each student and their teacher in the analysis carried out, from my analysis it is necessary to define what micro-historicity means to me (see next section, 10.3.1).

10.3.1 Defining a traced pathway in the actions, based on an enactivist approach

In the context of the mathematics classroom, each student, teacher and their experiences are part of a mathematical world in which interactions are occurring, showing our unique ways of acting. As a consequence, mathematics knowledge is unique in its shape, emerging through the historicity of interactions from the teacher and their students (in this study) whose actions are generating that particular moment of interaction (based on Varela, 1999, p. 7; Depraz, Varela &Vermersch, 2002, p. 156).

From these interactions, observable mathematical changes in the historicity of the students (or teacher) can be noted. Those changes that can be observed are determined by each student or teacher in this study (based on structural determinism, see chapter 3, section 3.9.3, p. 46). This means, that, although students (or teacher) are interacting, any change is according to the autonomy of each one, that can be shown through the decisions taken, in the moment, to act (based Varela, 1994). This may explain the different pathways of solving a mathematical problem that can be observed in the interactions between the students and also between the teacher and their students, as described in the example of solving equations (see chapter 3, section 3.9.4, p. 50)

In this context of behavioural changes, and through the process of my first observation of the transcript, there are different pathways that lead the observer to what the students or teacher are doing mathematically.

These unique pathways, chains of contributions in the interactions of each one, in a series of interactions with others, and can be observed by an observer through the decisions made for each one, because they are linked with their own historicity and their autonomy when they are doing mathematics.

A micro-historicity, to me, means a set of interactions that can be observed in the transcript in the pathway of each person, through the patterns and decisions made by the person.

Finally, what have I noted in the re-observation of the interactions between the students and their teacher and amongst the students themselves? In the next sections I will describe these observations.

10.3.2 Becoming aware

In the process of re-observing the data, I started to notice patterns of behaviour, particularly, a process of becoming coherently aware amongst students and a suspending moment between the teacher and their students. I could not observe the process of becoming coherently aware between the teacher and their students.

In the beginning, I named the *suspending moment* between the teacher and the student, *interval of waiting* (Ramirez, P., 2017b; Ramirez, P., 2018). From now on, with a goal of maintaining the same language and coherence in my writing, I will use the word *suspending* instead of interval of waiting. To note the suspending moment, I observe, in the actions performed by the participants, a distinction between a background and a new start in the interaction. This distinction between the background and a new start happens when there is a change in the micro-historicity of interactions of each participant.

To introduce this pattern, let me present the next episodes, which illustrate actions observed in the process of becoming aware mathematically through the interactions with the other students or teacher in the mathematics classrooms.

10.3.3 Suspending moment between teacher and their student[17]

In the mathematics lesson, groups consisting of a maximum of four students were solving the budgeting modelling task[18].

> A game manufacturer would like your team to create carnival game with 5 water bottles and 5 beanbags. Water bottles cost $1.00 each and beanbags cost $1.25 each. The game will be played at a carnival. The company expects 175 children to play. Small prizes cost $0.50 each, medium prizes cost $1.00 each and large prizes cost $3.25 each. The company plans to charge $250 for the game (including the prizes) and they want $100 profit, so you have $150 to spend on each game and the prizes. Plan how to spend your budget and use mathematics to show that your plan for the game will work for 175 children. Make a poster with:
>
> - a drawing of the game
> - the rules of the game
> - Extension: If you had to revise your budget to $100, what would you change? Justify your answer.

[17] The next analysis is part of my paper, Re-analysis of observations of lessons of students in Chile working on mathematical tasks, in Curtis, F. (Ed.), *Proceedings of the British Society for Research into Learning Mathematics* (November 2017).

[18] This mathematical task is part of a collection of mathematical modelling problems from the book *Guidelines for Assessment & Instruction in Mathematical Modeling Education (GAIMME)* (2016), Consortium for Mathematics and Its Application (COMAP), Society for Industrial and Applied Mathematics (SIAM), p. 129.

The original text of this problem was written in Spanish (see appendix 9, p. 284).

C1 S1: How [will] I know [how] to divide the prizes? [addressing the teacher] could be a way to change this sentence? [show how your plan for the game will work for 175 children].

C2 T: You make the game.

C3 S1: And if we buy only smaller prizes, we can stay within our budget.

C4 S2: The restriction says [the game] costs $150. We cannot exceed that amount. It must be exact.

In terms of the students' contributions, contribution 1 is the first action that I observed, which was a question from a student, S1, to the teacher. Later, the teacher replied (contribution 2). After that, student 1, who started with a question related to correcting the sentence, made a shift, a change from the original state (the question), and moved on to a suggestion related to the mathematical problem, "and if we buy only...". The shift in the actions in from contribution 1 allowed me to observe a change from the original state, which was the first question (contribution 1), to the other suggestions that were observed (contribution 3).

When the new start begins, with a suggestion (contribution 3), the first question or the *first start* is *transformed* into the *background context* because the student is no longer referring to correcting the sentence; he/she has moved on to talking about the smaller prizes. The background is important because it allows me to see if there is another change in the action and, therefore, if another distinction could be made by me as an observer.

By observing the last two contributions of the transcript (C3 and C4), I am seeing that the two students are engaged in the same actions, speaking about the budget and the restriction, maintaining their coherent behaviour, literally, of doing mathematics. I have observed no change between these two actions, therefore, there is no distinction between a new start and another background.

Observing the change in the actions when a teacher and the students are doing mathematics, for me it is possible to distinguish between any start in discussing mathematics and the background by taking into account, "the coherence and the capacity to understand details of how the discourse unfolds as a complex weft of multiple voices" (Towers & Martin, 2015, p. 252). This will be a way to begin to identify details about the actions that are observed (on the basic and subordinate levels), which could show patterns of behaviour, acknowledging how the students and their teacher were doing mathematics as seen through my observations.

Let me return to the modelling task transcript. As a first approach to these actions or "basic-level" observations, characterised trough my interactions with these actions, it is possible to observe that a student asks a question related to the prizes and another about some possible changes to a sentence in the task (C1), which may require answers. Thus, we could possibly see the action as questions and answers or another observation.

However, if I go into more detail and re-observe the interactions between the teacher and the students, the students' questions can, perhaps, generate a closed answer from the teacher or other students. Nevertheless, the reply that happened was a sentence, from the teacher, of "You make the game" (C2).

After that sentence, "You make the game", I observed a *suspending moment*, a space provoked by the suggestion and the new start (C3). In this suspending moment, the sentence stated by the teacher can trigger multiple answers, one of which was the student's reply: a conditional suggestion, with the use of 'if', to the other students that were participating in the dialogue (see C3).

Contribution 3, the new start, does not generate an interval of waiting between contribution 3 and contribution 4; there is no evidence of a shift between these two actions (C3 and C4). The answer from student 2 (C4) was a closed answer referring to the problem, "the restriction says..."; maybe this was due to the use of the conditional, "if", in the intervention (C3).

This situation, an interval of waiting, shares similarities with the suspending moment (Depraz at al. (2000) and Varela (2000) in chapter 6, section 6.2, pp. 92-95 *awareness and behaviour changes*), because, as the authors mentioned, the suspending moment is to remove yourself from your habitual actions in order to engage in another experience.

I observe, for example, student 1, moving from talking about changing the sentence and knowing how to split up the prizes, in contribution 1, to a mathematical condition considering the budget in the activity (C3). This change in his/her action, removing the doing to another experience, opens up another possibility for solving the mathematical problem. Within this pattern of behaviour, in the *interval of waiting,* the student moves from what he/she was doing to another action.

Following the pathway of S1 in his/her own historicity, deleting the other's contributions, but bearing in mind that this could happen, I can observe the shift in S1's action, which was triggered by the teacher's question, generating the suspension moment for S1, removing his/herself from what he/she was doing previously.

C1 S1: How [will] I know [how] to divide the prizes? [addressing the teacher] could be a way to change this sentence? [...show how your plan for the game will work for 175 children]

C2

C3 S1: And if we buy only smaller prizes, we can stay within our budget.

C4

For S2 and the teacher, it is not possible to remove the actions and observe the shift because, as an observer, I cannot see the unique pathways in the action if I only observe their contributions.

I will present another transcript. As part of the start of any mathematics lesson, the teacher and their students have an agreement to write the day in numerical terms according to the mathematical theme of the week. For example, for 23th June, 2017, one of the students suggested writing the date as $2^4 + 2^3 - 2^0 / 2^2 + 2^1 / 2^{10} + 2^{10} - 2^5 + 2^0$. After writing that, the teacher asked the students to check if the operations were according to the day.

After that intervention, the next dialogue followed:

C1 (4: 40_4:47) S1: Teacher, I would like to ask (/) if two to the power of ten plus two to the power of ten [$2^{10} + 2^{10}$] is equal to two to the power of eleven [2^{11}]?

C2 (4:49_4:55) T: Yes...I agree with you, yes, but could $2^{10} + 2^{10}$ be 2^{11}? Hold on a second (/), Why [would this be the case]?

C3 (4.56_5.15) T: Look [at] pupils, the proposal here (//) was what he/she asked me, *is it possible* that $2^{10} + 2^{10}$ equals 2^{11} (//). I said yes, but actually I want him/her to tell us why he/she can do this.

C4 (5.17_5.32) S1: Because, I have (/), because my (//), 1024 plus 1024 is [equal to] 2048 and two to the power of eleven is equal to 2048.

After many contributions between the teacher and their students, including S1, this interaction ended with the teacher explaining on the whiteboard, $2^{10} + 2^{10}$ equals 2^{11} using counting of 2^{10} (2 times 2^{10}) and later the property of powers for multiplication of equal bases: $2^{10} + 2^{10} = 2 \times 2^{10} = 2^1 \times 2^{10} = 2^{11}$.

As previously discussed, (chapter 6, section 6.1.1, pp. 78-80), through my interactions with the transcript, as an observer at a basic level of observation, this transcript could be organised as a series of questions and answers between a teacher and a student. However, if we look closely at the details, inspired by a "superordinate level" (Rosch, 1978, chapter 6, section, 6.1.2, pp. 80-84) of looking, conceptualising what I am seeing, the first start from this action (C1) generated another question from the teacher, "... $2^{10} + 2^{10}$ could be 2^{11}?" (C2). In contribution 2, after the question, "Why [would this be the case]?", I observed what I call an *interval of waiting* or *suspending moment*, which is a space triggered from the question pursued by the teacher, that makes something 'new' for the student, S1. The question could generate different re-actions, such as, another reply from the S1, or, as in the case of what happened here, a new start, a shift from the initial state (now background) which was the question from student 1 (C1), to a new start, with the potential for learning, which was the question from the teacher (C3).

This shift allowed me to observe the other question from the teacher, which sought to know more about the proposal made from the student at the start of this action (see C1). Contribution 3 does not generate an interval of waiting, given the features of the question, which seeks to obtain a direct answer from the student in order to explain the mathematical concept.

Following S1's pathway from his/her own micro-historicity and deleting the other's contributions, whilst knowing that the interactions with S1 involved the teacher, I can note the shift for S1, moving from the question to the teacher to their own answer. As an observer, this allow me to see, again, a suspension moment of S1.

C1 (4: 40_4:47) S1: Teacher, I would like to ask (/) if two to the power of ten plus two to the power of ten [$2^{10} + 2^{10}$] is equal to two to the power of eleven [2^{11}]?

C2 (4:49_4:55) T:

C3 (4.56_5.15) T:

C4 (5.17_5.32) S1: Because, I have (/), because my (//), 1024 plus 1024 is [equal to] 2048 and two to the power of eleven is equal to 2048.

S1 is removing his/her self from what he/she was doing previously, from the mathematical question to the teacher to his/her own answer, maintaining coherent behaviour in the action. He/she is still working on the equivalence of the exponents ($2^{10} + 2^{10} = 1024 = 2^{11}$).

10.3.4 Process of becoming coherently aware amongst the students

Re-observing the conversations between the students, I started to note changes in the interactions amongst students, leading in different directions in their own historicity and for me given how enactivism conceives cognition, this would be a way in which the students or teacher could start to be triggered by such differences in order to maintain their own mathematical action or let go of habitual actions and learn something new in the interactions with the environment.

This micro-historicity for me naturally implies observing coherent behaviour in the actions of each student or teacher (mentioned in Chapter 3, section 3.9.4, pp. 47-51) because each one builds or creates their own way to act, triggered by the stimulus from the environment.

I will present now three different transcripts that illustrate becoming coherently aware. I have named the pathways S*n*, in which S*n* is S1, S2, S3 and so on.

Pathway S6

In illustrating S6's pathway, I recognised that the students in the class had not studied any particular strategy to solve the equations within the academic term (that I observed). I observed three students working on solving the following equation:

$$x - \frac{5}{2} = \frac{5}{8}$$

The dialogue below followed:

C1 S2: Why should we use 10/8?

C2 S6: Look, I multiplied by four [on the denominator]. So, I got 8, and 10 minus 5 equals 5, so for that reason, [the value] is 10/8.

C3 S2: But, that is not right. You multiply by one of them and not both [numerator and denominator]. Look, it is 20/8 then the value [of x] is 25/8 [because 25 minus 20 equals five]. Should be both [numerator and denominator].

C4 S6: How is that?

C5 S2: I am speaking about to *amplify* [this is transliteration from Spanish meaning multiply numerator and denominator of the fraction by the same number].

C6 S6: Like this [shows his or her calculations to S2].

C7 S3: Hey, but how? How did you get the eight [speaking to S1]? Ah, true; it is in each one [writes the numbers in his notebook].

In the beginning, observing at the basic level, these interactions amongst these three students would be questions and answers. If I am going close to details inspired by observing at the subordinate level, I can note S2 is concerned about their classmate's answer in the equation (C1). S6 (C2) is explaining what he/she has done and S3 seems to be paying attention to what is happening between the other classmates (S6 and S2) because of the type of question asked in contribution 7, which also shows the structural coupling between S3 and the other students.

To observe the pathway for S6 and therefore the micro-historicity in the interactions, I will remove the other students whose are participating in this interaction.

C1 S2:

C2 S6: Look, I multiplied by four [on the denominator]. So, I got 8, and 10 minus 5 equals 5, so for that reason, [the value] is 10/8.

C3 S2:

C4 S6: How is that?

C5 S2:

C6 S6: Like this [shows his or her calculations to S2].

C7 S3:

As a consequence of removing S2 and S3 from the interaction (but bearing in mind that they participated in this), I can visualise a shift in the interaction of S6. In contribution 2, S6 was explaining what he/she had done to the equation for getting the result 10/8. However, after an intervention from S2, S6 makes a shift in what he/she was doing before. I can note the shift for the type of question, How is that? That question opens S6 up to new possibilities in their history of interactions. Later in contribution 6, S6 shows what he/she has done to S2 saying "Like this". Because of that I can observe, how the question "how is that?" from the student S6 is finalise with the shift of what she/he has solved mathematically previously.

This interaction ended with a change of procedure and answer to the equation by S6.

Pathway S4

Now, I will present another transcript in which I can observe the process of becoming aware mathematically in the interactions of three students. The conversation took place in a video-recorded lesson, when the students (aged 13–14 years old) were solving a budgeting task to create a carnival game with five water bottles and five beanbags. The students needed to spend $150 for the following items: bottles, beanbags and three kinds of prizes (small, medium and large). All the items had different prices, as 175 children are expected to play the game. There were three students working in a group, S4, S5 and S6.

C1 (6.43_6.45) S4: We must have 175 [children playing]; if not I won't play. [referring to she will not create the carnival game]

C2 (6.45_6.47) S5: Okay, // but it [the result about the distribution of the children] cannot be recurring [decimal number].

C3 (6.48_6.48) S6: Yeah, but if …

C4 (6.48_6.52) S5: She has a recurring number [as a result]; this doesn't work like that. We have a better chance of making [the game] without a recurring [decimal] number.

C5 (6.52_6.55) S6: Shh. Do you remember when teacher told us what happens if? Shhh.

C6 (6.54_7.01) S4: But I have to buy 58 [prizes] because 58 plus 58 plus 58 is 174.

C7 (7.01_7.03) S5: And where is the number three? And where is the recurring [decimal] number.

C8(7.01_7.05) S6: Shh. What happens if it is one hundred and seventy … *listen to me, listen*.

C9(7.05_7.10) S4: That is 175 exactly. But I have 174, which is not exact. So, it is not 175 [sound from snapping her fingers].

C10 (7.11_7.14) S5: So // we cannot do it with a recurring [decimal] number.

C11(7.14_7.22) S4: But not because with 58.333 [multiply by three] you have 175 [children], but 58 times three //

C12 (7.22_7.22) S5: Fifty-eight times three . . .

C13 (7.22 _7.23) S4: … is 174, and it is not a recurring [decimal] number.

C14 (7.25_7.31) S6: Look, what happens if all the children won the big prizes? We will *exceed* the budget *a lot*. Do you see it? //

C15 (7.34_7.34) S4: Let's have a look.

C16 (7.35_7.35) S5: So we cannot do it [dividing] by three [groups].

C17 (7.35_7.36) S6: There is a chance that all the players could win the big prizes; so [sound of hitting his/her hand on his/her desk] I suggest, the game [that we are creating] should have a trick.

C18 (7.37_7.42) S4: As the way I said it? No, I think we [must] divide by three [groups] that is 58 [players] each and we don't exceed the budget because it is 174, not 175, and if we exceed [the budget] we'll sell tickets [to play the game]. It's not so complicated.

Considering the perturbations received by the others, and therefore explored explicitly and in detail in my analysis, this is S4's own pathway of becoming aware mathematically.

I am not observing how awareness could be formulated amongst students because “interaction is not instructive, for it does not determine what its effect are going to be” (Maturana and Varela, 1992, p. 96) but how it emerged by some stimulus received (i.e., a question) through interaction with the world.

In order to state clearly and in detail what I observed in S4’s interaction with his/her peers, his/her unique trajectory, I will re-structure the interaction removing the received perturbations but bearing in mind that this interaction took place under a history of interactions between S4, S5 and S6. The sequence can be condensed as follows:

C1 (6.43_6.45) S4: We must have 175 [children playing]; if not I won’t play. [referring to she will not create the Carnival game]

C2 (6.45_6.47) S5:

C3 (6.48_6.48) S6:

C4 (6.48_6.52) S5:

C5 (6.52_6.55) S6:

C6 (6.54_7.01) S4: But I have to buy 58 [prizes] because 58 plus 58 plus 58 is 174.

C7 (7.01_7.03) S5:

C8(7.01_7.05) S6:

C9(7.05_7.10) S4: That is 175 exactly. But I have 174, which is not exact. So it is not 175 [sound from snapping her fingers].

C10 (7.11_7.14) S5:

C11(7.14_7.22) S4: But not because with 58.333 [multiply by three] you have 175 [children], but 58 times three //

C12 (7.22_7.22) S5:.

C13 (7.22 _7.23) S4: … is 174, and it is not a recurring [decimal] number.

C14 (7.25_7.31) S6:

C15 (7.34_7.34) S4: Let’s have a look.

C16 (7.35_7.35) S5:

C17 (7.35_7.36) S6:

C18 (7.37_7.42) S4: As the way I said it? No, I think we [must] divide by three [groups] that are 58 [players] each and we don't exceed the budget because it is 174, not 175, and if we exceed [the budget] we'll sell tickets [for play the game]. It's not so complicated.

In the trajectory of interactions with S4, I can observe the pathway of becoming aware for his/her, which, for me, is related to suspension, redirection and a moment of letting go, of becoming aware (based on Depraz, Varela and Vermresch (2000, 2003) and Varela (2000)) mentioned in chapter 6, section 6.2, pp. 92-95.

Looking at each of these aspects in turn, firstly, suspension is removing our actions from what we are doing, avoiding the prejudices and habitual actions of one's current knowing as doing and therefore opening one's mind to new possibilities of actions (based on Depraz *et al*., 2003, p. 25).

I observed that S4 was habitually showing what she/he had done to S5 (after her recurrent perturbations of it not being possible to work with recurring number) and S6 trying to mention a restriction about the budget and the potential winners as shown in the first specification on my observation. As evidenced in contributions C1, C6, C9, C11, C13, the usual habits of S4 relate to dividing the participants of the game into three groups, saying that she is not working with recurring numbers.

The moment of redirection is the shift in our attention from the "exterior" to the "interior" (Depraz *et al.,* 2003, p. 25) via suspension, which leads to, for instance, emerging events, content, patterns and gestures from whence we can find the new view (Varela, 2000, p. 5). In this context, considering the observation of interactions between the participants, redirection is an action that would be noting when S4 is moving to a different action, recognising what she has done before, but noting at the same time, something new in the action performed. As an observer/knower, I noted, after contribution 13 and before contribution 15, where there is a moment of awareness that emerged through the prompt received by S6 (C14) who says, “what happens if all the children won the big prizes? We will exceed the budget a lot…” that leads to redirection saying “let’s have a look”, an instance of becoming aware. This expression, “let’s have a look”, shows to me how S4 is moving to ‘something’ new, and I, as an observer/knower, noted this new seeing as part of her trajectory in her history, perceiving this new awareness after the perturbation received from S6, opening to the possibility to see other mathematical aspects in his/her work.

However, in contribution 18, S4 stayed where the interaction started for his/her, divide by three groups, without moving his/her actions in this interaction into a new action, which can be expressed by the expression, “No, I think we [must] divide by three [groups] that is 58 [players] each” (C18), recognising what she has done before, habitually. At the same time, I can observe that S4 recognises a suggestion made by S6 because she said, “we don’t exceed the budget because it is 174” (C18) noting that something new has been triggered by the action *redirection.*

S4’s action shows that, although she/he became aware, acting differently in his/her trajectory, she/he decided (a change in his/her action) to stay in his/her normal mathematics habit, for example staying in the action of dividing into groups of three players.

This situation is not surprising because *letting go* is a moment of decision in which a person can follow the same action in which the interactions start or accept difference and move on into a new action. This action is defined by acceptance, which is triggered by the quality of the attention of each one, showing their autonomy in the decision to take it. In this sense, a person is actively paying attention while reflecting (based on Depraz *et al.,* 2003, p. 47).

Pathway S1

The next transcript details the beginning of the lesson in which the students are solving the mathematical modelling task, described previously in section 10.3.4. Basically, the students are speaking about the next part of the following problem:

> The company plans to charge $250 for the game (including the prizes) and they want $100 profit, so you have $150 to spend on each game and the prizes

C1 S1: Ah, so we need to see how many bottles we will buy and how many bean [bags]?

C2 S3: Yes, spending [the money] the prizes, if we should spend…

C3 S1: Ah, and we should recover the money with the selling of the game for $100.

C4 S3: The value of the game plus $100. They want to earn $100 when they sell the game at $250.

C5 S1: Ah, so it must be $150 plus $100, so they want to earn $250.

C6 S3: No, with their $250. They already are earning the $100. Look, this is what they have to buy, and this is what they want to sell.

C7 S1: But they get $250, and that is what they will earn, the total

C8 S3: No, here it says “The company plans to charge $250 for the game, including the prizes” [reading the problem].

C9 S1: That’s why I am saying $150 to spend.

C10 S3: And how much did they say they would sell the game for?

C11 S1: I don’t know.

C12 S3: $250.

C13 S1: They didn’t say that they would charge $250.

C14 S3: [The company] plans to charge $250.
[Silence]
C15 S3: First line.
C16 S1: Ah, indeed.
C17 S2: It is better to ask the teacher.
[The teacher is coming.]
C18 S3: It is just saying that they want to earn $100 per game.
C19 T: Read the problem as many times you need to.
C20 S1: Each game that the children will be playing.

At the *basic level* of my observations this transcript could be characterised by a simple discussion about how much profit the company wants to earn, as shown by "they want to earn $100" (C4); "they want to earn $250" (C5); and "they want to earn $100 per game"(C18).

Observing in detail what this conversation says about mathematics, at the beginning, S1 is speaking about buying the bottles and bean [bags] to design the game (C1), saying, "How many bottles we will buy and how many bean [bags]". The conversation moves to the action of recovering the money that they spent on the items (prizes, bean bags and bottles), with the word "recover" (C3). S1 says that to recover the money means "selling of the game for $100" (C3). However, S3 clarifies to S1 about selling the game "plus $100". "They want to earn $100", which is evident from contribution 4. Thus, it is possible to observe, until this point (contribution 3), S1 wants to sell the game for $100, but not to make a profit as show the other student S3 (on contribution 4).

In this conversation about mathematics, the first shift of action concerning S1 I can observe happens during contribution 5. S1 follows the idea of "earning", triggered by S3 saying, "They want to earn $100" (C4). S1 says, "so they want to earn $250" (C5). For student S1, the $250 is composed of $150 plus $100. Now, she/he is no longer speaking about selling the game at $100. S1 wants to earn $250 (C5). S3 replies, "with their $250. They already are earning the $100" (C6). In C7, S1 says, "But they get $250, and that is what they will earn, the total".

Although S1 is moving toward the idea of earning, it seems to be that she/he is seeing "earn $250" (C5) and "sell the game at 250" (C4) as a similar kind of earning. When S3 asks S1, "How much did they say they would sell the game for?" (C10), S1 replies, "I don't know." (C11).

Later, in contribution 16, there is a second shift in the action of S1; she/he accepts to sell the game at $250. S1 is no longer speaking about earning $250.

I will, now, remove the contributions of the other participants in this mathematical episode with the goal of following the traced pathway of S1.

C1 S1: Ah, so we need to see how many bottles we will buy and how many bean [bags]?

C2 S3:

C3 S1: Ah, and we should recover the money with the selling of the game for $100.

C4 S3:

C5 S1: Ah, so it must be $150 plus $100, so they want to earn $250.

C6 S3:

C7 S1: But they get $250, and that is what they will earn, the total.

C8 S3:

C9 S1: That's why I am saying $150 to spend.

C10 S3:

C11 S1: I don't know.

C12 S3:

C13 S1: They didn't say that they would charge $250.

C14 S3:

[Silence]

C15 S3:

C16 S1: Ah, indeed.

C17 S2:

[The teacher is coming.]

C18 S3:

C19 T:

C20 S1: Each game that the children will be playing.

Following the traced pathway of S1, it is clear that she/he stops speaking about earning $250 (contributions 5 and 7). S1 takes a moment, including a silence (between contribution 13 and 16) *suspending* what she/he was doing before, *redirecting* his/her action and noting the 'new' (selling for $250 and earning $100) when she/he says, "Ah, indeed" (see C16). I can note that, in the expression, "Ah, indeed", there is a moment of *letting go* and accepting the plan "to charge $250" (S3 in C14) when they are making the game. S1 now follows the sentence of another student S3. In contribution 20, S1 says, "Each game that the children will be playing", after S3 says "It is just saying that they want to earn $100 per game" (C18).

The following is the traced pathway of S3.

C1 S1:

C2 S3: Yes, spending [the money] the prizes, if we should spend…

C3 S1:

C4 S3: The value of the game plus $100. They want to earn $100 when they sell the game at $250.

C5 S1:

C6 S3: No, with their $250. They already are earning the $100. Look, this is what they have to buy, and this is what they want to sell.

C7 S1:

C8 S3: No, here it says, "The company plans to charge $250 for the game, including the prizes" [reading the problem].

C9 S1:

C10 S3: And how much did they say they would sell the game for?

C11 S1:

C12 S3: $250.

C13 S1:

C14 S3: [The company] plans to charge $250.

[Silence]

C15 S3: First line.

C16 S1:

C17 S2:

[The teacher is coming.]

C18 S3: It is just saying that they want to earn $100 per game.

C19 T:

C20 S1:

I observe that S3 wants to sell the game at $250, as shown in contribution 4 ("they sell the game at $250"); contribution 6 ("with their $250"); contribution 12 ("$250"); and contribution 14 ("plans to charge $250"). And she/he also considers the profit of the game, as shown in contribution 4 ("They want to earn $100"); contribution 6 ("They already are earning the $100"); and contribution18 ("it is just saying that they want to earn $100 per game").

Therefore, for S3, I do not observe a moment of redirecting, suspending or letting go of his/her doing in mathematics; she/he is doing the difference of earning and selling in the problem and explaining this situation to S1.

10.3.5 Each historicity triggers inter-actions in the mathematics classroom

10.3.5.1 *A historicity of the teacher*

Through observing a mathematics classroom, I noted the teacher constantly questioned the students, and this can be evidenced in the transcriptions described in chapter 9, section 9.4, pp. 171-182. For example, I observed the students working on changing both terminating and recurring decimals into fractions. They were using a procedural technique that is common in the Chilean context:

$$0.\dot{5} = \frac{5-0}{9} = \frac{5}{9} \qquad 2.\dot{5} = \frac{25-2}{9} = \frac{23}{9} \qquad 2.\dot{5} = 2\,\frac{5-0}{9} = 2\,\frac{5}{9}$$

The idea behind this procedure is working with the limit when *n* approaches infinity of a given geometric series, for example:

Given the geometric series $\sum_{i=1}^{n}\left(\frac{5}{10^i}\right)$, if I calculate the limit of that series as follows:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{5}{10^i}\right)\right)$$

$$\lim_{n\to\infty}\left(\frac{\frac{5}{10}\left[\left(\frac{1}{10}\right)^n - 1\right]}{\frac{1}{10} - 1}\right)$$

$$\frac{\frac{-5}{10}}{\frac{-9}{10}} = \frac{5}{9}$$

Then, the next equivalence is $0.\dot{5} = 0.555\ldots = \frac{5}{9}$

I did not observe this procedure used with the geometric series as part of the lesson, and this may be because, according to the Chilean curriculum, teaching about geometric series is reserved for 17–18-year-old students (Ministerio de Educación de Chile, 2009, p. 239).

At the beginning of the lesson, as shown in the next transcript from my fieldnotes, the teacher asked questions concerning how to change both terminating and recurring decimals into fractions. One of the students, S5, replied:

> C1 S5: I remembered something with zero. For example, in $2.6\dot{4}$ you write 26 above and then minus 6 (referring to the operation on the numerator $\frac{26-6}{90}$).

C2 T: Why did you decide on this number?
C3 S5: Because I remembered something with zero.

Considering C2 as an example of what happens within the mathematics classroom, I observed a question from the teacher to the student, which was followed by another question from the teacher regarding how to change recurring decimals into fractions.

The students engaged in other mathematics questions or replies (C1). C3 is evidence of how the student, S5, replied to the teacher's follow-up question.

In another lesson, there was a problem to solve about bacteria:

> En una observacion de bacterias se cuentan 3 bacterias. Transcurrida media hora de la primera observación se cuentan el doble de bacterias, media hora después se cuentan nuevamente el doble de bacterias. Si se espera que el comportamiento de la población de bacterias crezca siempre de la misma forma, responda cuantas bacterias hay a) 1 hora, b) 2 horas, c) 2 horas y media, d) 3 horas, e) 4 horas y media, f) 8 horas, g) 1 día.

[Text translated by the researcher from Spanish to English. The original text does not mention "scientist"; however, this term has been added to give coherence to the translation.]

> A cultivation of bacteria began with three of them, in the first observation made by the scientist. After a half hour, he counted the bacteria twice. Half an hour later, he counted the bacteria twice again. If the scientist expects the bacteria population is growing at the same rate, then how many bacteria will there be after a) one hour, b) two hours, c) two and a half hours, d) three hours, e) four and half hours, f) 8 hours and g) one day.

The next conversation occurred when the teacher and the students began checking their work on the bacteria problem:

C1 (3.07_3.19) T: Before starting with other questions [referring to the word problem], what action implies double or the amount being doubled?
C2 (3.20_3.21) S1: Multiply.
C3 (3.21_3.22) S2: Multiply by two.
C4 (3.23_3.27) T: Multiply by two. Did anyone use another strategy instead of multiplying by two?
C5 (3.27_3.28) S3: I added it.
C6 (3.28_3.28) S4: Me too.
C7 (3.28_3.29) T: What did you add?
C8 (3.30_3.47) S3: Like this, for example, if there are three bacteria at the beginning, then for the first half hour, I had to add three plus three. Then, for the next half hour, I had to add the results obtained before plus *that result again.*
C9 (3.47_3.58) T: Yes, for example, at the start, you said that there are three bacteria, and then after *the first half hour*….
C10 (3.58_3.58) S5: Six.
C11 (3.59_4.01) T: According to you, three plus three.
C12 (4.01_4.01) S3: Yes.
C13 (4.03_4.05) S6: Teacher, are we starting from there?
C14 (4.05_4.05) T: *Either*, which is the other way?

If I remove the student interactions, following the teacher's historicity in this interaction becomes easier:

C1 (3.07_3.19) T: Before starting with other questions [referring to the word problem], what action implies double or the amount being doubled?
C4 (3.23_3.27) T: Multiply by two. Did anyone use another strategy instead of multiplying by two?
C7 (3.28_3.29) T: What did you add?
C9 (3.47_3.58) T: Yes, for example, at the start, you said that there are three bacteria, and then after *the first half hour*….
C11 (3.59_4.01) T: According to you, three plus three.
C14 (4.05_4.05) T: *Either*, which is the other way?

The teacher asks the students multiple questions, as shown in contributions 1, 4, 7, 9, 11 and 14, that link to the teacher's own history of interactions in mathematics education.

In addition, as show in contribution 7, I noted that the teacher is trying to address what student S3 has done. Later, in contributions 9 and 11, the teacher is following up on S3's answer, until contribution 14, when the teacher invites the student to explain his/her work.

The teacher asking students questions is evidenced through the interactions with the students and also in the interview and in the teacher's historicity. In the chain of interactions mentioned in chapter 5 (section 5.2, pp. 67-71), it is possible to observe the importance of the teacher's historicity as part of the classroom dynamic.

Each teacher has a unique way to act in the mathematics classroom and the teacher's historicity shapes this uniqueness, i.e., the questions noted in the teacher's own past interactions with others are shown explicitly in the teacher's actions.

The action of asking questions triggers classroom interactions with and between the students. In the transcript above, regarding the bacteria problem, the students reply to the questions, as shown in contributions from S3 (contributions 5, 8 and 12).

The usual way the interviews were carried out was with the teacher (T) and the researcher (I) sitting around a table in the teacher's room (see figure 19).

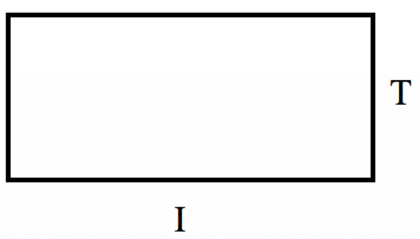


Figure 19: Position of the teacher and researcher within the interviews.

In the fifth interview with the teacher, which started with an open question about her history with mathematics, I explored the action of asking questions linked with her own historicity of interactions, as shown in the next transcript:

> T: He [referring to her schoolteacher] always took notice and asked me questions, and I felt he understood what I was saying. And I reckon one of the things I highlighted from him is the dialogue […] he made the lesson based on questions that generated dialogue.

Notably, this historicity recalls different moments in the teacher's life, including the teacher's childhood and undergraduate studies. The teacher, in her own history, makes a distinction that she named *dialoguizante*, which means using dialogue the same way her university teacher did when she was a student, as shown in the next transcript:

> T: He [referring to her undergraduate teacher] had this thing *dialoguizante* for saying in some way. He was good at writing [on the witheboard] like me. […] He had the idea of writing and asking. He finished the activities completely and always made examples or asked questions.

From her undergraduate studies, the teacher evoked her university instructor in her interactions and questioning.

The third interview concerned solving the mathematical modelling task (mentioned previously in chapter 10, section 10.3.3, p. 202) of how to allocate a budget to create a carnival game for 175 players with five water bottles and five beanbags and to also award three kinds of prizes: small, medium and large. I observed how the teacher started to consider the big prizes to be an important factor in how to solve the problem, as shown in the next interview extract:

C1 T: I'm thinking of the possibilities, considering the money that I have and the cost, but these are the same five bags. I need to check. Let's suppose that 175 [players] drop everything, then maybe I don't have enough [money] to give prizes to everyone. Would be fantastic if I did not have a limited budget, because now I need to be bound to the budget that I have. Then 175 [children] and $3.25 for the big prize [multiplied on the worksheet].
C2 I: That is $568.75.
C3 T: $568. 75 can't be all of them.
C4 I: All of them winning, no other option.
C5 T: Then, if my goal is to access the big prize, of course, then how can I reduce this? Plus, I need to decrease the $150 that I will spend on the supplies, minus five and then minus five times $1.25, this means $145 minus $6.25, or $138.75.
C6 I: [A total decrease of] $11.25.
C7 T: Look.
C8 I: Not in all of them.
C9 T: Right.
C10 I: Mm-hmm.
C11 T: $138.75 for prizes.
C12 I: Then, our budget was not $150.
C13 T: No, it was $138.75. Perfect. Now, I need to think in some way of these 175 [players]. Let's think, on the first instance to see the possibilities of the small prizes, this $138.75 I will divide into only big prizes.
C14 I: Ah-ha.
C15 T: I haven't finished how many prizes of each one I will buy, but I'm imagining that all the prizes will be by team, and the first team to access the rule, that I don't know yet, will have the big prizes.
C16 I: Then it will not be an individual [game], it will be by a team.

I observed that the teacher was solving the mathematical modelling problem, stating: "Then 175 [children] and $3.25 for the big prize [multiplied on the worksheet]" (C1), emphasising working with big prizes to divide the money according to the budget allocated and the players. This can also be evidenced when the teacher says, "if my goal is to access the big prize" (C5) and also when considering the option of the small prizes but bearing in mind the big prizes (C13).

Similarly, in the interview held with the five students after the lessons that involved solving the mathematical modelling problem described previously, they mentioned the situation of working with the big prizes.

To me, the consideration by the students of working with the big prizes is according to the question that the teacher asked in the classroom, which triggered interactions between the students in which they considered the big prizes when solving the mathematical problem, as shown in the following extract from the second interview with the students:

> C1 S5: We started [solving problem] with 175, and we never thought all [the participants] would win the big prize, but <u>the teacher tells us</u> <u>what happens if all win the big prize.</u> So, there [referring to the moment the teacher spoke with them] <u>we had to change the game [about] how to make it with 5 groups, and [the game] that is fixed [the game], and we do not exceed the budget.</u>
> C2 I: And 175 in five, where is the five coming from?
> C3 S5: Because it was [referring to 175] divisible by 5.
> C4 I: As it is divisible by 5, we can make it. Can you use other numbers, let's say, 3?
> C5 S2: But [175] is not divisible by 3.
> C6 I: And did the question the teacher asked at the end of the lesson work?
> C7 S3: We were questioning a lot [what the teacher said] because in all the things we had done, we did not think on that because we had done an [equal] number of small, medium and large prizes. But then the teacher said, <u>what happens if all [the participants] won the big prizes,</u> and then, from there we started questioning the entire [mathematical] problem. <u>We realised we had left things behind, so we started to do everything again.</u>
> C8 I: And in your case, because you started working alone and then with her, what happened [when you were solving the problem]?

The interactions triggered by the teacher had similar results in the students, with the questions triggering more questions and focusing the students' attention on different aspects of the problem.

As evidenced in the extracted interview above, student S3 considered what the teacher had said about big prizes, mentioning that "the teacher said what happens if all [the participants] won the big prizes?" (C7). Similarly, S5 mentioned "the teacher tells us, what happens if all win the big prize" (C1). The teacher's question, pointing out the big prize, generated interactions between the students. S3 and S5 decided to change their approach to solving the mathematical problem, with S5 expressing "we had to change the game [about] how to make it with 5 groups, and [the game] that is fixed [the game], and we do not exceed the budge" (C1).

The teacher expressed that questions are an important part of the students' actions because they generate dialogue and subsequent actions, as evidenced in the following interview extract:

> T: In the group, you are observing [referring to me as the researcher], for example, it allows a lot of dialogue, and even though they want to speak a lot, there is an attitude to question [your] methodology [referring to the type of mathematics taught]. Instead, other courses [if I am doing the same] don't work like that, they want to make exercises, the formula and do it, five hundred times the same exercises. In this course, I can start with a question to motivate and generate a dynamic. Instead, in the other, I need to start with the [mathematical] definition, blah, blah, blah and then do the exercise […] I feel in this course, they are more sensitive, and in general, we can do more things when the dialogue is generated. The other course is more traditional, more structured, and the students do not go as deeply into the subject [mathematics].

Finally, the historicity of the teacher about questioning is in line with one of the ways that she is doing mathematics with her students. Noting for example what is important to her mathematically (i.e., the big prizes in the mathematical modelling problem) and applying this idea through the questioning in her mathematics classroom. Doing that she is generating inter-actions that are going in a similar way with their students, allowing them to maintain coherence of behaviour of the students. The students make a change in the action about how to solve the problem, but they are still solving the problem.

10.3.5.2 A historicity of the students

Using x instead in a historicity of the students

In the first interview with the group of students (after they had solved a mathematical modelling problem related to trapezoid tables, see chapter 8, section 8.9.2, p. 142), I asked them about something that I had observed in the classroom. The students were solving a mathematics problem, with the teacher in their classroom, about counting the bacteria in a population after a certain number of hours. The teacher asked, "How many bacteria will there be in *n* hours?" Later, the teacher wrote on the whiteboard, $3 \cdot 2^{2n}$. The teacher was stood in front of the whiteboard.

The next conversation is before the teacher wrote $3 \cdot 2^{2n}$ on the whiteboard.

C1 (37.35_37.43) T: Let's suppose the number of hours is any // *n number* of hours, *n*

C2 (37.44__37.45) S1: That could be an equation?

C3 (37.45__37.58) T: An expression, because we are not looking for the ~unknown number but *an expression* // What do I need to write here to calculate in the *softwar*e the number of bacteria? What is coming *now*?

C4 (37.55__38.00) S2: $3 \cdot 2^{20}$

C5 (37.45__38.13) T: Three times two is *how many?* Three by the action of multiplying, what exponent? The expression will be used for two hours, seven hours, twenty- four thousands hours.

C6 (38.07__38.07) S2: $3 \cdot 2^{20x}$

C7 (38.11__38.11) S2: Times x

C8 (38.17__38.17) T: In any case, that is a *n* of number, *n* number, *n* hours

C9 (38.22__38.25) T: What was the exponent for two hours?

C10 (38.25__38.26) S2: Double, double of …

C11 (38.27__38.28) S3: Double of n

C12 (38.28__38.31) S2: *n*// twice

C13 (38.28__38.31) T: Double of a number, and if it is with half hour will be 0.5 [hours] it works the same, *perfect* [The teacher writes on the whiteboard $3 \cdot 2^{2n}$]

In the first interview with the group of the students, I asked them about the mathematics classroom, in which they were involved with the bacterial problem mentioned above. This extract is from the first interview with a group of four students. The method of choosing the participants in the interview was according to the frequency of interactions that I observed in the classroom. The students who frequently interacted with the teacher, or with their peers, or working by themselves or with the teacher and their peers. The students (S1, S2, S3 and S4) and the researcher (I) were sitting around a table in the library (see figure 20):

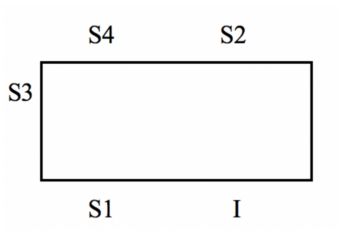


Figure 20: Position of the participants in the first interview.

C1 I: This is something about mathematics. Tell me, for example, about when you saw the power. I remember you said to your teacher to write the *x*, do you remember that?

C2 S2: Write the *x*?

C3 I: Yes, when the teacher had [written something] two to the power of *n* [2^n], and you had two to the power of *x* [2^x]. What did you mean by that?

C4 S2: Ah, <u>for me, that is usual.</u>

C5 S3: That <u>the unknown value is the *x*.</u>

C6 S2: The unknown [value] is the *x*.

C7 S3: It can be another letter, but for us, *x* is the letter.

C8 S1: Yes, we use that [referring to *x*].

C9 S2: It is our maña ['maña' is a Chilean term used when the behaviour is a habit].

C10 S1: I remember that a teacher told me, I am new here [referring she/he started this year in her current school]. I remember that a teacher told me that I had to write a dot[19] not an x, because it was believed when we are growing up [as students] that it [the *x*] was not for multiplication.

C11 S2: But we are speaking about the unknown value.

C12 S3: By definition, the *x* is the unknown [number].

C13 I: And what does it mean? Yesterday somebody wrote the *x* [referring to the bacterial problem].

C14 S3. That is any number, and it is the rule. For example, writing three exponent *x* or three exponent n is the same. For us, the *x* is more common, seeing the *x*

The historicity of this students, S1, S2 and S3 do what is common for them to do when they "do not know the value of something", which is evident from extract from the interview when student S2 says, "for me, that is usual (C4) ; the unknown [value] is the *x*, (C6) "it is our maña" (C9); S1 says "we use that [referring to *x*]" (C8) and S3 "*x* is the unknown [number]" (C12).

S3 says, "the unknown value is the *x*"(C5), "it can be another letter, but for us, *x* is the letter"(C7), "by definition, the *x* is the unknown [number]"(C12). S2 and S3 share the idea that *x* is the unknown number.

Although the teacher is introducing a new letter to identify a variable or the unknown number in the interaction with the students, these students decide that the

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[19] In the Chilean context usually the use of the symbol "·" means multiplication (i.e., 7 · 8 means 7 x 8). However, in the early years of teaching mathematics, it was more common to use x to represent multiplication.

letter should be *x*, which makes mathematical sense to them based on their own histories with the use of *x*. It is usual for them, a maña; they have the habit of using *x* instead of another letter. This situation about the use of *x* for multiplication is similar to what happens in contributions 6 and 7 of the transcript of the classroom interaction. The student, S2, says to the teacher "$3 \cdot 2^{20x}$" (C6) and "Times *x*" (C7) although the teacher is writing "*n*" as the variable.

A historicity of comparing mathematical problems

In the second interview with the group of the students, they mentioned their approach to the mathematical modelling problem and how this contrasted with their usual way of working with mathematics problems. The students (S1, S2, S3, S4 and S5) and the researcher (I) were sitting around a table in the library (see figure 21).

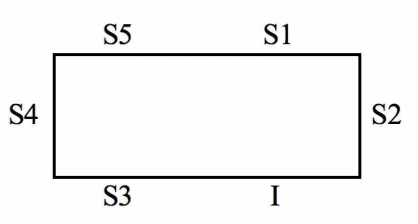

Figure 21: Position of the participants in the second interview.

An extract of dialogue from the interview follows:

C1 S1: In addition, there's another thing I would like to say, in history or language, there can be more than one answer to a question, but in mathematics, there is only one answer because it is like that.

C2 I: And that is what happened with this problem?

C3 S4: No.

C4 S5: No.

C5 I: But that is also a mathematical problem.

C6 S1: This problem is more like […]

C7 S5: This problem [deleted for coherence] <u>has different ways of solving it, but there are mathematical problems that [the answer] is always the same for all of us.</u>

[…].

C8 S4: But in here all the things are together. <u>You cannot compare because always there is a better [answer].</u>

C9 S1: With this problem, a lot can change a lot because [deleted for coherence] the person who will buy the game is telling you that you must use the three prizes and that is the variable for the prices used. <u>Here, you have the freedom of choosing how you want to make the game</u> because they said you are the enterprise that makes the game.

C10 I: And that is different from what you have done in mathematics?

C11S1: Mostly, <u>maths is problems and exercises</u>, but some of them are open like this.
C12 S3: I like the problems.
C13 I: What do you mean by exercises? [addressed to S1]
C14 S1: <u>Solving problems.</u>
C15 S3: <u>Problems with data that are long and extensive.</u>
C16 S2: Data.
C17 S3: And you can follow [the procedure] <u>instead, here you need to analyse.</u>
C18 S2: I like <u>the others more.</u>
C19 S3: <u>That are</u> more mechanical.
C20 I: Do you like the others more?
C21 S2: Equations or things like that I like.
C22 S3: <u>This is not so mechanical because</u> you need to highlight [what the problem says]
C23 S1: In this one, you can do as you wish. Instead, when you have equations, you have rules of how to do it; first you do this, then this. <u>Instead, here</u> you can start from whichever part you want.
C24 S5: <u>It is creativity of each one of us. You can solve something that looks not fixed.</u>

The students started to compare their usual actions of doing mathematics with solving mathematical modelling tasks; they see links with solving mathematics problems. They use words such as ‘here’, ‘instead’, ‘like this’, ‘that are’, ‘the others’ and ‘this is’ as shown in the transcript. When S1 said, “With this problem, a lot can change a lot […] <u>Here, you have the freedom of choosing how you want to make the game</u>” (C9), S3 says, “you can follow [the procedure] <u>instead, here you need to</u>

analyse" (C17), and also contrasting, "That are more mechanical" (C19), with, "This is not so mechanical […]" (C22). S2 says, "I like the others more"(C18). S5 notes, "It is creativity of each one. You can solve something that looks not fixed" (C24).

All of the students start to compare actions. In this comparison, the distinctions noted from the students are adding to what actions differ in the mathematical modelling problem process as compared with doing usual problems. Collecting together what each student says:

S1

> in mathematics, there is only one answer because it is like that; Here, you have the freedom of choosing how you want to make the game; Mostly, math is problems and exercises, but some of them are open like this

S3

> Problems with data that are long and extensive; And you can follow instead here you need to analyse;That are [referring to solving problems] more mechanical; and, This is not so mechanical because you need to highlight [what the problem says].

S2

> I like the others more;Equations or things like that I like.

S5

> it is creativity of each one of us. You can solve something that look not fixed

As the students discuss solving the mathematical modelling problem, it is not surprising that they differ in their approaches, since each one is autonomous and makes their own decisions (based on an enactivist approach). This is also evident in the previous transcript, from interview 1, when they speak about using letter x instead of n.

The usual way of solving problems for the students is linked with each one's history of interactions. S1 notes "the freedom" (C9) in the mathematical modelling problem, contrasting her/his actions when solving a usual mathematical problem as "in <u>mathematics, there is only one answer because it is like that</u>" (C1). S5 says, "It is creativity of each of us [referring to the mathematical modelling task].You can solve something that looks not fixed" (C24); From S5's comments, it is possible to infer that the usual problems that he/she solves are fixed. Usual problems are connected with the types of interactions that he/she is doing currently in his/her historicity, allowing her/him making a distinction about problems that are fixed and not.

In a similar way, S3 notes that usual problems for her/him are "Problems with data" (C15), and, in the other mathematical modelling problems, "you need to analyse" (C17), which shows his/her action (analysing) in the problem. For S2, the action of solving the equations is what he/she likes.

10.3.6 Mathematics empathy through the interactions of:

10.3.6.1 A teacher and their students[20]

In the recursive actions performed by students and teachers, when there is a shift in their actions, a distinction is noted by the observer. I am analysing the transcription using the third stage of empathy proposed by Depraz and Cosmelli

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[20] The next analysis has been presented in my paper, Empathy in interactions in a grade eight mathematics classroom in Chile. In J. Golding, N. Bretscher, C. Crisan, E. Geraniou, J. Hodgen & C. Morgan (Eds.), *Research Proceedings of the 9th British Congress on Mathematics Education*. (pp. 143-150) (see list of publication appendix 1, pp. 270-271).

(2003), as I mentioned previously (chapter 6, section 6.4, pp. 100-101), which is intrinsically related to a person's interactions in his/her mathematics world.

The following text presents a conversation, between a teacher and a student, that was observed in a lesson. In this lesson, the class was solving operations associated with calculating the square root of a number, which sometimes involves solving an equation. However, I recognise that, within the time that I observed, the students had not been working on any particular strategy to solve such an equation.

This episode of interactions has been chosen because it provides an account of patterns I also observed with other students in this classroom.

The following transcript begins with a question from a student, which is directed at the teacher, regarding solving the equation $\sqrt{16 \cdot 81} = 4x$. The point in the equation has been left, because, as I explain early in the interview described in section 10.3.5.2, in the Chilean context the use of the symbol "·" means multiplication, i.e., 16 · 81 means 16 x 81.

C1 (17.22_17.25) T: Let's see; what have you done? [deleted for coherence] // What did you get?

C2 (17.26_17.26) S1: The square root of 81 [the student wrote in his/her exercise book √9].

C3 (17.28_17.31) T: That is nine, not the square root of nine // because you said the number multiplied by itself gives [81].

C4 (17.41_17.43) T: Not [that], because the square root [of nine] is three.

C5 (17.43_17.51) T: Are you understanding? This is the process. It's not necessary to follow operating in square roots. What is the square root of 81?

C6 (17.52_17.52) S1: Okay, so . . .

C7 (17.53_17.54) T: What is the square root of 81?

C8 (17.55_17.55) S1: Nine.

C9 (17.56_17.56) T: Done it; it's not the square root [of nine].

C10(17.58_17.59) S1: And then, for example, aha! Here, I can't follow because it is wrong.

C11 (17.59_18.01) T: Mmm

C12 (18.02_18.04) S1: Then there, // could be it? [referring to $\sqrt{25 \cdot x} = 35$.

C13 (18.06_18.07) T: What is the square root of 25?

C14 (18.07_18.10) S1: Ah, no; I could have //. I got this $\sqrt{81} = \sqrt{9}$; $\sqrt{625} = \sqrt{25}$; $\sqrt{36} = 6$.

C15 (18.12_18.23) T: Of course, because what you are doing is a square-root chain. This means that, what you get, you calculate the square root again. Is it asking which number, when multiplied by itself, gives 625?

C16 (18.23_18.25) S1: Five.

C17 (18.25_18.27) T: Twenty-five, not the square root of twenty-five.

C18 (18.28_18.33) S1: No, not yet because I had written the number with the square root.//

C19 (18.43_18.48) T: It was the [square] root of 625; you got 25. [Interruption: Another student asks the teacher if he can go look for something that he needs.]

C20 (19.03_19.03) T: [Square] root of 625 is . . .

C21 (19.05_19.05) S1: Twenty-five.

C22 (19.07_19.19) T: So, it is equal to 25. That is asking again, and you're not writing the answer to your question, ~ what is that number that multiplied by itself gives 625 ~ [this number is 25].

C23(19.21_19.28) S1: Ah // I've multiplied twice; for example, here [referring to what she had written before C14].

C24 (19.30_19.32) T: That is alright, // delete the square and ask yourself again.

C25(19.32_19.35) S1: Oh! // done it!

C26 (19.36_19.36) T: What is the number that multiplied by itself gives 36?

C27 (19.37_19.38) S1: So, the same here.

C28 (19.40_19.40) T: Exactly//

C29 (19.42_19.42) S1: OK, alright.

In the transcript, although the structural coupling is happening all the time, because the participants are interacting with others in this mathematics classroom, I observed in particular a structural coupling between the teacher and this student in the dialogue having the teacher say, “multiplied by itself gives [. . .]” in contributions 3, 15 and 22, triggering an action, mathematically, that can be evidenced later in the answers provided by the student in contributions 8, 16 and 23. Similarly, the same happened when the student replies to the question about what the square root of 625 was (in contribution 21), given that previously the teacher told the student the answer to the square root of 625 (as shown in contribution 19).

I observed that when a change was triggered by a chain of events, within the micro-historicity of interactions, the action increased. In contributions 10, 14 and 18, the student manifested his/her understanding of the situation of what she/he was doing saying, “it is wrong” (C10), but being more specific about his/her mistake, “I had written the number with the square root” (C18), which means she/he has written the square root of nine as the result (C14). Later, I noted other distinctions between what happened before those actions related to the mistake of the square root and contribution 23. This distinction, and the shift in the action, were triggered by the teacher’s intervention in contribution 22. The student finds a way to make sense, as shown by contribution 23, “I’ve multiplied twice”. I must recognise that the student receives other stimuli by the teacher before contribution 22, for example, comments such as “square-root chain” in contribution 15, and including, “What is the square root of . . .” in contributions 5, 7 and 13. However, after contribution 22, the student is now showing what she/he has noted as evidence (contributions 23, 27) or more specifically he/she manages to make sense, as shown in contribution 23.

In addition, in contribution 1, when the teacher says, “Let’s see; what have you done?”, this shows empathy, leading to a possible interpretation of what the student has completed. The teacher starts the interaction from the action of the student, showing empathy that could be related to an interpretative understanding of difference (Depraz & Cosmelli, 2003). Later, the teacher makes explicit her interpretation of what the student had done in contribution 15 with the phrases, "what you are doing is a square-root chain"; "you calculate the square root again"; and in contribution 19, "you got 25". The teacher interprets what the student is doing, which is "leading with the possibility of understanding" (Depraz & Cosmelli, 2003) by mentioning what was done previously in their mathematical interactions. A consequence of this kind of interaction is recognition (or noting), based on the teacher's interpretation, of what the student has accomplished mathematically.

This kind of empathy, which is manifested through my interpretations, was evident in other episodes in other lessons (see the transcript below p. 239) from my field notes of an observation regarding the definition of recurring decimals in fraction conversions). The *italic* words indicate what I have noted in the transcript related to empathy.

> Teacher: It is not [pure] mathematics language; it is colloquial [language]. But, *it is for your understanding.*

The phrase, "it is for your understanding", shows empathy based on the interpretation of the teacher regarding the students (based on Depraz & Cosmelli, 2003). A definition written in colloquial language could generate understanding.

In addition, in the sixth interview with the teacher, one of the aspects addressed was her mathematics teaching history,

> *I know which questions* I have to ask to trigger the content in some way. I'm encouraging my students [referring to students from 8 grade] so that *they can connect and do* with the knowledge that was there, which I assembled and re-appropriated[...]

The sentences, "I know which questions I have to ask [. . .] the content" and "I'm encouraging [. . .] connect and do", can be associated with the teacher and her view of doing mathematics in her classroom. Her interpretations and understanding of what happen in the classroom also shows her historicity (also mentioned in section 10.3.6). This expresses that she is part of the classroom (structurally coupled) but also reaffirms empathy based on the interpretations mentioned above.

10.3.6.2 Empathy between students

The following text presents another conversation that was observed in the same lesson described above between two students. S1 asks the other, S2, about the square root. The conversation is as follows,

C1 (28.00_28.00) S1: Is it the [square] root? [The student has written in his/her notebook next procedure]:

$$\sqrt{16} \cdot \sqrt{x} = 32$$
$$4 \cdot \sqrt{x} = 32$$
$$4 \cdot \sqrt{8} = 32$$

C2 (28.05_28.11) S2: It is. What are you getting for the [square] root [referring to the 8 under the square root] if you know that four times eight is equal to 32? Then this [referring to *x*]//~, what must it be?

C3 (28.12_28.12) S1: [Deletes [$4 \cdot \sqrt{8} = 32$ in his/her notebook and then writes $4 \cdot \sqrt{64} = 32$]

C4 (28.17_28.19) S2: That's right. // You have done it.

In C2, S2 makes a distinction with S1 saying, "What are you getting for the [square] root". In this distinction, the phrase, "What are you getting" shows how S2 expresses mathematical empathy, "leading to the possibility of understanding" (Depraz & Cosmelli, 2003, p. 173) with S1. The expression "are you", in the sentence "what are you getting", which is said by S2, illustrates "the recognition of the other's experience as belonging to the other" (based on Thompson 2001, p. 6, see chapter 6, section 6.4, p. 100).

This empathetic position from S2 to S1 can lead to an understanding or a misunderstanding of the situation (based on Depraz & Cosmelli, 2003, see chapter 6, section 6.4, pp. 100-101) about what S2 has done when he/she solved the equation:

$$\sqrt{16} \cdot \sqrt{x} = 32$$

In addition, in C2, the type of intervention made by S2 can trigger other actions in S1, which is not surprising because S1 and S2 are a structurally coupled (based on enactivist theory, see Chapter 3, section 3.9.2, pp. 45-46) in their mathematics classroom.

In C2, it is not possible to say which part of the intervention, let's say, "What you are getting for the [square] root [referring to 8 under the square root]", or "if you know that four times eight is equal to 32, then this [referring to x]", from S2 to S1 provokes the change in the following action from S1.

I observe a shift in action emerging from S1, because at the beginning, S1 has an expression that for him/her represents an equation $4 \cdot \sqrt{8} = 32$, but later, after the intervention, S1 deletes what he/she has written and moves on to another expression $4 \cdot \sqrt{64} = 32$. This action from S1 (triggered by the intervention of S2) allows S1 to solve the equation (see chapter 5, section 5.3, pp. 71-73, *learning as coherent behaviour*), which is reflected when he/she writes $4 \cdot \sqrt{64} = 32$.

10.4 Metaobservation

When the interactions take place between a teacher and the students and amongst the students themselves, inspired my enactivist approach, their unique way of acting is informed by the autonomy in their decisions (see chapter 3, section 3.5, pp. 30-31), which I observed through the shifts in their performed actions.

The three 'lenses' employed in the data (emotions expressed through empathy, the process of becoming aware and actions of distinctions) that I bring as an observer to these observations (the micro-historicity and historicity of them) allow me to learn of the emergence of mathematics learning, through the shifts observed in the actions (which is learning, see chapter 5, section 5.2, p. 71). I must, therefore, trace the chains of actions of each person bearing in mind the input they are receiving from others. For me, this tracing has been established as a common characteristic in my observation of the data, which is independent of the persons who are interacting (teacher or students). Doing so, I can observe the moment when the students become aware of their empathy for one another and when they recognise each other's history.

The interviews with the participants help support the details that I noted through observing the interactions of the teacher with the students as well as amongst students themselves. They provide the opportunity to knit together how the learning about mathematics that I observed emerges through the actions in the mathematics classroom.

CHAPTER 11: Discussion

11.0 Introduction

In this chapter, I present my last metaobservation, which is a re-observation of my previous metaobservations. Looking back at my experience, learning again as an iterative process as part of my way of conceptualising my doing, I noted the concept of *equifinality* as a way to arrive from different roads to the emergence of learning through conversation about mathematics.

I explain my contribution to knowledge through methodological themes such as the importance of historicity and re-observing as an analytical tool. I discuss how this approach could be adopted by others, including the use of working with transcripts that shows interactions through conversations. Also, I answer my research question (section 11.5.1, pp. 251 - 252) about how the emergence of mathematical knowing through conversations, from my own observations, happens, describing the limitations of my study and adding some further research.

11.1 My last metaobservation

11.1.1 Experiencing looking back

How did I start this research? I was intrigued by the way we learn mathematics, particularly how we acquire proficiency in a classroom setting where teachers and students are present. This specific focus (i.e., teacher and students present in a mathematics classroom) captured my attention given that I used to be a mathematics teacher. Looking back, I recall the experience of taking action to shape the manner in which students interacted with others. Such shaping happens because learning is interaction (see chapter 5, section 5.2, pp. 67-71). For example, the encounters of a mathematics teacher with her own instructors at different stages of life influence how she acts in the present, raising questions or *dialoguizante* (see chapter 10, section 10.3.5.1, p. 225). This experience is important for any teacher's classroom.

I have been observing the historicities (including micro-historicities) of a teacher and her students, taking account of the autonomy of each person, who bring to the exchange their own decisions. With these interactants being born and reborn in each interaction, a unique interaction with mathematics learning emerges. This process allowed me, as an observer, to see the shifts in learning, noting, for instance, the moment at which a student becomes aware of what is happening (for example, with respect to a budgeting problem, see chapter 10, section 10.3.4, pp. 211-220) and what kind of perturbation is triggered.

11.1.2 Learning again

When I began this thesis, one of my metaobservations relates to how I found myself learning mathematics with my students, as well as how I became a part of their environment and how they became a part of my surroundings. I distinguish between environment and surroundings because, as I previously explained (see chapter 3, section 3.7, pp. 36-38), surroundings refer to our local interactions with others, whereas environment pertains to a set of surroundings. I can say that I was part of the students' environment, given that interactions occurred in this setting, but I cannot claim to have been a part of the surroundings of each individual since belonging depends on the interactions that one has with different people.

In the process of carrying out this research, I found myself learning mathematics again. As I acquired an understanding of how mathematics learning emerges from students and teachers, I observed that I was re-learning the subject in another way. I was paying attention to details in the conversations between the research participants, noting the particular changes in each person with respect to her/his historicity and how he/she builds up his/her actions. This situation, from an enactivist point of view, is unsurprising, owing to the experience of learning and the nature of learning as interaction. I created my own surroundings with the decisions (a type of interaction) that I took in this entire process. The observations, unstructured interviews, reading, sharing my study with the research community, observing videos and writing allowed me to engage with and expand my thoughts about the emergence of mathematics learning through conversations.

11.1.3 Iterative process

Within my own historicity as a researcher, I was learning through the interactions in which I engaged as I carried out this study. Deriving an accurate account of my observations, regarding the emergence of learning amongst the participants, necessitated access to the richness of the details that characterise the behaviours of the participating teacher and students and their interactions through mathematics conversations. Another essential component was the iterative process (chapter 8, section 8.8.3, pp. 137-140; chapters 9 and 10), in which I was truly involved. This involvement motivated me, in this final metaobservation, to conduct a re-observation and then return to my writing. The iterative process also corresponded with my reflexivity as I conducted the research; moving backward and forward, I recognised the pathway that I needed to tread, the attention that I devoted to my learning in the beginning, my approach to categorising the actions of the participants (chapter 9, section 9.3, pp. 159-166) and how I moved on to re-observing the data.

The reader may wonder why I chose an iterative process. As described in the methodology section (chapter 7, section 7.2, pp.106 -108), Reid (1996) asserted that an enactivist researcher works under different perspectives. This definition, for me, is an invitation to re-observe the interactions that a researcher has with a given study, with a view to cultivating a comprehensive understanding of what has been observed and noting not only the researcher's own learning but also the emergence of learning amongst research participants (chapter 10, section 10.3, pp. 198-242). Using varying standpoints as guidance, showing in what ways observation has been carried out, thereby bringing transparency to the process of research and enabling an observer to know what is being observed.

11.2 Equifinality

My intention as a researcher, in engaging with the collected data, has been explorative. I did not want to find 'average' behaviours amongst the participants, nor did I try to fix the data, for example, with *epochè* (see chapter 6, section 6.2, pp. 92-95) I understand that the mathematics classroom is a place where different interactions happen between participants. As mentioned previously (chapter 5, section 5.2, pp. 67-71), Davis noted that "one thought sparks another, and an idea spreads through the room; knowledge in this setting seems to exist in and consist of the participants' patterns of interaction" (1995, p. 4).

The principle of equifinality "means that the same results may spring from different origins" (Watzlawick, Bavelas, & Jackson, 1967, p. 108). I took different paths in carrying out recursive observations to determine what happens during interactions between teachers and their students and those occurring amongst students through my observations. Attentively surveying conversations, I took account of the shifts in actions in the historicity of each of the participants. By paying attention to the chain of interactions in the mathematical episodes chosen for my analysis under varied routes (i.e., emotions, awareness and distinctions), I interconnected all these engagements through the observed actions of the participating teacher and students and what the students were doing in the mathematics classroom, specifically through the conversations that they took part in during the examined mathematics episodes. Interconnection was also supported by my consideration of the interviews with some of the participants in the selected mathematical lesson.

Following the idea of equifinality, the three lenses—emotions, awareness and distinctions—cleared the way for me to comprehend how mathematics learning begins to arise in different scenarios.

11.3 The importance of the historicity

A chain of interactions shapes our approach to observing the mathematics classroom. Let me discuss an example of a recent and brief historicity of my own. I was a teacher's assistant in a course about advanced qualitative methods for postgraduate students (master and doctoral levels). In endeavouring to highlight the role of an observer and what aspects are observed, I showed students three minutes of the beginning of a mathematics lesson, which I monitored in this doctoral research. I then asked the students to tell me what they observed, assuring them that they do not need to worry about the Spanish audio because this would not prevent them from carrying out their observations (all the attendees of this course spoke English, and two of them spoke both English and Spanish.). What did they observe? The students provided different responses, such as, the observation that the teacher must be ready to leave given that she is wearing a coat in the room. The respondents also stated that they do not wear their coats inside a room. I explained that, in Chile, classrooms usually do not have central heating and that this video was taken in winter. Other students called attention to the structure of the chairs, which were arranged one by one. One student said, "I used to sit in my school similar to the current situation, with tables positioned together, and around it, a group of students are chatting". One of the Spanish speakers pointed to what the mathematics teacher said in the explanations, which reminded her/his of her/his mathematics teacher.

I am telling you these stories because, for me, they share a common factor. What is this commonality? All of them demonstrated that the mathematics interaction amongst the students related to their own historicities. In this study, different views arise from the historicity of the mathematics teacher and the relevance of her raising questions to the students and the historicities of the students in the interviews when they compared conventional mathematics with a mathematical modelling problem. This difference stems from their unique ways of interacting and performing mathematics.

Thus, once again, our historicities (chain of interactions) determine our way of interacting, which happens naturally because historicity is a composite of interactions (based on chapter 3, section 3.3, pp. 27-29), and interactions trigger the distinctive manner in which the teacher or students in this study conducted themselves—an influence expressed by the decisions that they took.

I argue that an exhaustive observation of the emergence of the mathematics learning process within interactions requires an enquiry into the historicity (Varela, 1999, p. 7; Depraz, Varela, & Vermersch, 2002, p. 156) of each participant; an orientation that includes historicity encourages a researcher to observe details and shifts in action when a conversation about mathematics takes place. In my case, exploring how each of the participants differed in terms of historicity enabled me to observe the similarities in the mathematics actions performed by the teacher and the students. Empathy could have led to the understanding or misunderstanding of a mathematical situation. The process of becoming aware mathematically began from the interactions of the participants, but an essential requirement was to note the particular awareness that is linked or not to the *suspending, redirecting and letting go moment* (Depraz, Varela, & Vermersch, 2000, p. 25). This step seems necessary as it enables a researcher to follow a participant's own pathway in interacting with others.

Historicity is part of us as individuals; we are constantly interacting and acting in our own world, which is shaped by the interactions that we have with others. To this engagement, we bring our unique way of learning and observing, and we are always performing our historicities and our chains of interactions. Finally, these multiple interactions (in the 'now') entail more learning, and what I want to say with respect to this issue is that learning is always evolving. As soon as an interaction starts, a new possibility of learning is happening and a pattern of behaviour can be seen. Within this emerging potential, a new historicity is created and, therefore, another possibility to observe the emergence of mathematics learning arises.

11.4 Re-observation as an analytical tool

11.4.1 Re-observation

Enactivism maintains that learning happens in interaction. Taking into account the manner by which learning is perceived, enactivism as a methodology invites me to think about interactions of interactions—an idea inspired by the definition given by Reid (1996) regarding learning about learning. Data revision, interactions of interactions—especially observations of the observations made by me, as an observer or, more simply, re-observation as an action, is, for my interactions, an analytical tool, which allows me to 'doubly' witness what I see in the emergence of mathematics learning through conversations. I, an observer, thus move from a descriptive account of actions to making explicit what I am observing.

This process of re-observation also contributes to a comparison with my initial evidence of what I observed, making explicit what I see, my own interactions with observed data, responding to the necessity of specifying what I learnt in the process of research. I note, for example, the shift in actions of research participants through their individual historicities and coherent behaviours, that is to say, maintaining a mathematical action or perhaps moving on to a new one.

This methodological approach could be directly adopted by other researchers, (or myself again) in other study. However, what comes from the observations in the study will be dependent on what is observed and the type of the interactions of the researcher. For that reason, I cannot say the researcher would have the same findings as me.

11.4.2 Observing pathway in the historicity of interaction

Through the re-observation, I put forward an analytical tool for observing pathways that connects to the historicity of each of the participants. This tool involves re-observing and writing about the interactions of only one person but bearing in mind his/her engagement with others (for an example, refer to chapter 10, section 10.3.2, pp.203-206), so that another can see the distinction, that I, as observer, am noting in the patterns of interactions with others (the teacher and students in this study).

This analytical tool could be adopted by other researchers who want to study the interactions of the participants, for example through conversations.

11.5 Observing the emergence of mathematical knowing

11.5.1 How mathematical knowing emerges through conversations about mathematics

Through my re-observation of the data, I found that the emergence of mathematical knowing, that is reflected in the details of shifts in interactions through conversations about the subject, is linked with the historicity of each of the participants. Their unique way of performing mathematics is characterised by empathy, through interactions between the teacher and the students and interactions amongst the students, which can lead to understanding or misunderstanding of a mathematical action. During development of awareness amongst the students, through the shifts in actions and the moment of suspension between the teacher and the students and amongst the students themselves, each historicity (from the teacher or the students) triggered interactions in conversations about mathematics, allowing for the observation of the focus of what the participants were doing. An example is the teacher's presentation of a question (chapter 10, section 10.3, pp. 198-242).

11.5.2 Limitations

Although I was looking for a school that was implementing the mathematical modelling curriculum, when the data was collected, the course that I was observing was not aligned with this part of the Chilean mathematics curriculum. This situation can be explained by the recent changes in the mathematics curriculum (see appendix 8, p. 282). In this same context about a mathematical modelling task, when the teacher worked with the task in her classroom, the extension question ("If you had to revise your budget to $100, what would you change? Justify your answer") was not used. The teacher told me that this was because she preferred to work with the other questions and that, "Time flies".

I studied the historicities of the students through their interactions in the classroom and through the interviews with them. For instance, I identified characteristics that were common to them when they solved a mathematical task. However, I did not follow the interactions of a particular student in the classroom the whole time and did not conduct an individual interview because such an approach does not correspond with the proposal of my study. I have studied interactions through conversations in an environment where a teacher and students are present. A limitation of this study could be that I did not consider the relationship of the body of each individual with the environment from which mathematics learning emerges, with respect to the manner in which the participants expressed themselves through their voices (a type of movement that involves the body). On the basis of my enactivist position, interactions with an environment entail multiple options for action. I concentrated my observations on only one of them, that is, mathematics conversations.

11.5.3 Further research

In my doctoral research, I probed into learning under three lines of enquiry: emotions through empathy, distinctions in actions and the moment of becoming aware mathematically. All of these, I noticed, are underpinned by the observation of historicity in interactions when a mathematical conversation occurs. Bearing in mind historicity and unique interactions that occur when mathematics learning emerges, I formulated the following two recommendations for further research:

- Exploring empathy through interactions amongst individuals of different age groups and varying genders of students and teachers.

I investigated empathy through mathematical conversations in an 8th-grade classroom (12- to 13-year-old students), which gave rise to the question of the extent to which we listen to one another in a mathematics classroom. An extension of this part of my research would be to look into other grade levels, such as, primary, high school or undergraduate. Doing this type of research, as a longitudinal study, may provide an extensive observation of behaviour noting how empathy through conversations, seen as interactions, is carried out in other ages and gender. Secondly:

- Bodily experience

In the field of embodied cognition, Lakoff and Nuñez (2000) emphasised bodily experience with mathematical concepts through metaphors. In this regard, some authors, such as Palatnik and Abrahamson (2018), delved into the rhythmic enactment of the body, being mindful of its relationship with quantitative reasoning and opportunities for implementing such techniques. The body of each individual is associated with the environment from which mathematics learning can emerge.

In my publication about empathy through interactions, I raised this issue about physical movement. Kardas and O'Brien (2018, p. 533) suggested that "[t]he more people watch others perform (without corresponding practice), the more they think they can perform the skill" Accordingly, I needed to ask whether empathy, expressed physically, can be enacted in mathematics for students and teachers to make sense of what they are doing. (Ramirez, 2018, p. 150). Doing this type of research supports doing mathematics through conversations and observing the relationships between body movement and conversations about mathematics.

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Appendices

Appendix 1: List of Publications

Although my PhD consists of developing a thesis without having published a paper, encouraged by my supervisors, Laurinda Brown and Rosamund Sutherland, I decided to work on publications about my research through attending conferences, with the goal of gaining feedback from the academic audience and therefore learning from them, challenging myself in my current thoughts and justifying these, while further disseminating my current research.

My publication record consists of the following:

A. (2018). Empathy in interactions in a grade eight mathematics classroom in Chile. In J. Golding, N. Bretscher, C. Crisan, E. Geraniou, J. Hodgen & C. Morgan (Eds.), *Research Proceedings of the 9th British Congress on Mathematics Education*. (pp. 143-150) (3–6 April 2018, University of Warwick, UK). Online at http://www.bsrlm.org.uk/wp-content/uploads/2018/11/BCME9-Research-Proceedings.pdf, accessed 27th June, 2019.

B. (2018). Shifts in mathematics interactions between grade eight students from an enactivist perspective. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5). (p. 143), Umeå, Sweden: PME.

C. (2017). Re-analysis of observations of lessons of students in Chile working on mathematical tasks. In F. Curtis (Ed.), *Proceedings of the British Society for Research into Learning Mathematics 37*(3), November 2017. Online at http://www.bsrlm.org.uk/wp-content/uploads/2017/12/BSRLM-CP-37-3-11.pdf, accessed 27th June, 2019.

D. (2017). Teachers' beliefs about mathematical modelling: An exploratory study. In T. Dooley, & G. Gueudet, (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10, February 1–5, 2017)*. Dublin, Ireland: DCU Institute of Education and ERME. Online at https://hal.archives-ouvertes.fr/hal-01933470/document, accessed 27th June, 2019.

Appendix 2: Ethics application approval

Ethics Online Tool: application signed off

[redacted]

Fri 13/01/2017 13:31

To: Paola Ramirez

Reply all

Inbox

Your online ethics application for your research project "Characterising the emergence of mathematical modelling components from enactivist perspective." has been granted ethical approval. Please ensure that any additional required approvals are in place before you undertake data collection, for example NHS R&D Trust approval, Research Governance Registration or Site Approval.

For your reference, details of your online ethics application can be found online here:

http://www.bristol.ac.uk/red/ethics-online-tool/applications/46644

Appendix 3: Ethical Discussion

SoE RESEARCH ETHICS FORM

It is important for members of the School of Education, as a community of researchers, to consider the ethical issues that arise, or may arise, in any research they propose to conduct. Increasingly, we are also accountable to external bodies to demonstrate that research proposals have had a degree of scrutiny. *This form must therefore be completed for each piece of research carried out by members of the School, both staff and students*.

The SoE's process is designed to be supportive and educative. If you are preparing to submit a research proposal, you need to do the following:

Arrange a meeting with a fellow researcher: The purpose of the meeting is to discuss ethical aspects of your proposed research, so you need to meet with someone with relevant research experience.

A list of prompts for your discussion is given below. Not all these headings will be relevant for any particular proposal.

Complete the form on the back of this sheet: The form is designed to act as a record of your discussion and any decisions you make.

Upload a copy of this form and any other documents (e.g. information sheets, consent forms) to the online ethics tool at: https://dbms.ilrt.bris.ac.uk/red/ethics-online-tool/applications.

Please note: Following the upload, you will need to answer ALL the questions on the ethics online survey and submit for approval by your supervisor (see the flowchart and user guides on the SoE Ethics Homepage).

If you have any questions or queries, please contact the ethics co-ordinators at: gsoe- ethics@bristol.ac.uk.

Please ensure that you allow time before any submission deadlines to complete this process.

Prompts for discussion: You are invited to consider the issues highlighted below and note any decisions made. You may wish to refer to relevant published ethical guidelines to prepare for your meeting. See http://www.bris.ac.uk/education/research/networks/ethicscommittee/links/ for links to several such sets of guidelines.

1. Researcher access/ exit
2. Information given to participants

3. Participants right of withdrawal
4. Informed consent
5. Complaints procedure
6. Safety and well-being of participants/ researchers
7. Anonymity/ confidentiality
8. Data collection
9. Data analysis
10. Data storage
11. Data Protection Act
12. Feedback
13. Responsibilities to colleagues/ academic community
14. Reporting of research

Be aware that ethical responsibility continues throughout the research process. If further issues arise as your research progresses, it may be appropriate to cycle again through the above process.

Name(s): Paola Ramirez G.
Proposed research project: Characterising the emergence of mathematical modelling components from enactivist perspective.
Proposed funder(s): Conicyt (scholarship from Chilean Government)
Discussant for the ethics meeting: Laurinda Brown
Name of supervisor: Laurinda Brown

Has your supervisor seen this submitted draft of your ethics application? Y/N
Please include an outline of the project or append a short (1 page) summary:

The aim of this research is to characterise the emergence of mathematical modelling components between teachers and students within a learning environment. To address this question, I adopt the enactivist perspective to look for a process of adaptation that involves the emergence of aspects of mathematical modelling through the interaction between teachers and students. I also examine the history of interactions that can be linked to the implementation of mathematical modelling as a cycle.

This research project will be carried out within the context of qualitative research, as the study's focus is on the emergence of the component of mathematical modelling; processes related to participants' adaptation, interaction and decision-making will be observed. The investigation will be conducted in three Grade 8 classes at different schools in Santiago, Chile. The participants will be approximately 90 students (aged 13–14 years) and their three teachers respectively (Ministry of Education of Chile, 2015). The data collection method will be unstructured interviews recorded by audio for three teachers and four students per school, within two phases - before and after the implementation of mathematical modelling within a lesson - observing each group in each school within a month and video- and audio-recording the lessons.
Ethical issues discussed and decisions taken (see list of prompts overleaf): This study will be carried out within three months Chilean schools and will include working with minors, children, and adults. With this in mind, we have

discussed the importance of access to all participants being based on honesty and the clarification of any doubts, as needed.

Research access to participants will be according to the requirements of the school (including the teachers) and parental consent, in the cases of the minors involved. We will access the participants in accordance with all school rules. We will abide by all ethical policies of the schools in question and Chilean law.

The information given to participants will be based on the research project and conveyed via letter, in the first instance, due the limitation of the distance between Chile and England. When I arrive in Chile to collect data, and if I find that that the teachers, parents or the community have any questions, I will respond according to the goals of my research. I will also suggest holding an information meeting, if they desire one, in order to clarify any questions. If community members ask questions beyond the scope of my research, such as about activities at the school or students' behaviour, I will respond that it is not possible for me to answer the questions because it is inappropriate and not the purpose of my research. I will also make it clear that teachers, the school community and parents representing the children have the right to voice any concerns or to complain about any issue during the data collection process. I will provide them with my email and the email address of my supervisor, should they wish to contact either of us after the data collection process is complete. I will make sure that all participants know that they are free to leave the project at any time, but I will suggest that, in such a case, I would appreciate a note telling me of their departure, in order to made the decision explicit.

Because the study will involve minors and teachers, it will be necessary to collect informed consent from a parent or legal guardian of each child and from all teachers. I will not start the lesson observations before obtaining the consent of all parents.

The data collecting process will be undertaken within school buildings, so all issues related to the safety and well-being of participants and the researcher are subject to the conditions of the school. I will ask for the schools' internal procedures for warnings for example in case of earthquake, fire or other cases, such as what to do if a student does not feel well during my time with him or her.

Data will be collected through observations and anonymous interviews; all participants will be assigned a coding number for use in field notes. I will explain to participants the importance of maintaining their anonymity as a way to protect their identity and the right to avoid any judgment. I recognise that the use of videotape recordings implies exposure, therefore I will suggest that participants' faces be blurred or otherwise altered through editing, and that the camera be placed behind the students.

During the interviews with minors, I will suggest that a person from the school be present as an observer, in order to maintain a good relationship with the school's procedures. I also recognise that this is a point of possible conflict, because I cannot control what the observed can say once the interview is finished, but I will remind everyone of the importance of anonymity as a part of the research process.

Confidentiality concerns are also an element related to the data collection process. For example, if someone asks me for my opinion about mathematics or behaviours, I will limit my answer and indicate that I cannot answer that type of question, so as to maintain the participants' confidentiality.

The data will be stored in the computer centre of the university and destroyed when my study is finished. All reporting in this study will be anonymous. If a school wants to share its name, I will discuss with its administrators what this implies in term of the participants' anonymity.

The feedback from my project will be shown to the participants, so they will know what I am doing in the research. They can also ask me to delete any portions with which they disagree.

If you feel you need to discuss any issue further, or to highlight difficulties, please contact the SoE's ethics co-ordinators who will suggest possible ways forward.

Signed: Paola Ramirez G (Researcher) Signed: Laurinda Brown (Discussant)
Date: 12/12/16 14/12/16

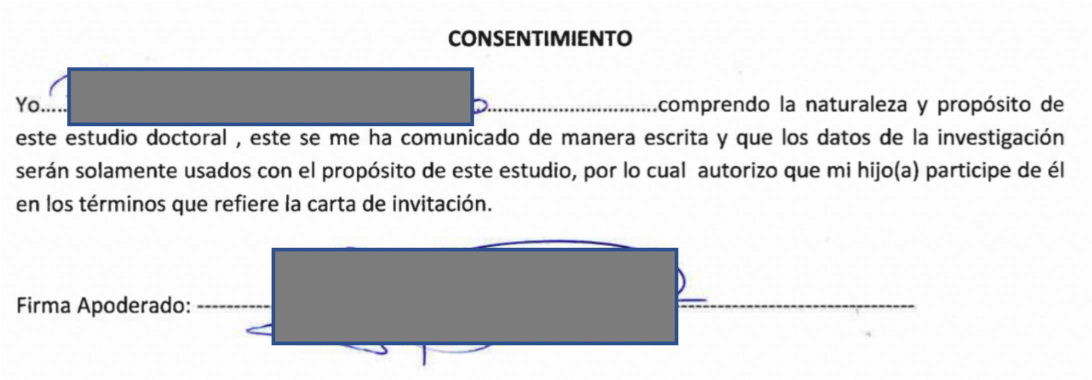
The next page details the information sheet that will be provided to the participants of my project. To write the information sheet, I have considered the example of guidance from the University of Edinburgh: title, invitation, explain what will happen, confidentiality/anonymity, time-consuming and future information.

For the consent form, I have used the same template with a few modifications from the research ethics consent guidance from the Faculty of Science Research Ethics, University of Bristol, because that form is for my kind of participants. This information sheet and consent form are examples of my future writing, because I am in the process of progression.

School of Education
Tel:

Doctoral Researcher: Paola Ramirez

Appendix 4: Example of consent form (Spanish version)



CONSENTIMIENTO

Yo.....[redacted]...............................comprendo la naturaleza y propósito de este estudio doctoral , este se me ha comunicado de manera escrita y que los datos de la investigación serán solamente usados con el propósito de este estudio, por lo cual autorizo que mi hijo(a) participe de él en los términos que refiere la carta de invitación.

Firma Apoderado: ---------[redacted]-------------------------------

Appendix 5: Example of consent form (translation from Spanish version)

I, ________________, understand the proposal of this doctoral study. This has been communicated with me by writing. I give my permission for the data from this research to be used for research purposes.

Parent signature: ______________________

Appendix 6: Mathematical modelling task

Brilliant

Many thanks for the prompt reply.

Best

Paola

Mrs Paola Ramirez G.
PhD Candidate
School of Education
University of Bristol
35 Berkeley Square
Bristol BS8 1JA

...

From: [redacted]@comap.com <[redacted]comap.com>
Sent: 29 April 2019 17:56:27
To: Paola Ramirez Gonzalez
Cc: Paola Ramirez Gonzalez
Subject: Re: reference your guidelines GAIMME

Hello

COMAP has no issue with reference the GAIMME report

Best

[redacted]

[redacted]@siam.org>

Reply all

Yesterday, 21:27

Paola Ramirez Gonzalez

Dear Ms. Ramirez:

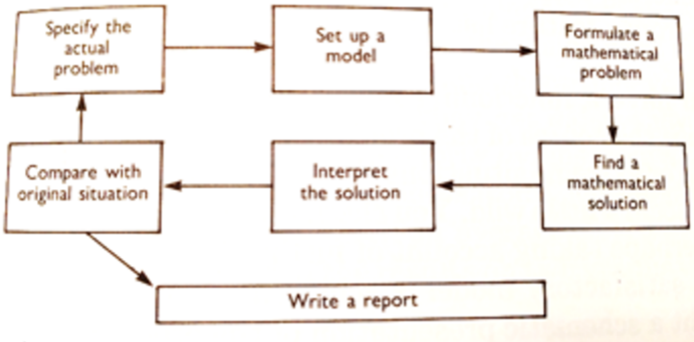
SIAM is happy to give permission to quote these two problems. Please acknowledge the original GAIMME publication.

Sincerely,

[redacted]

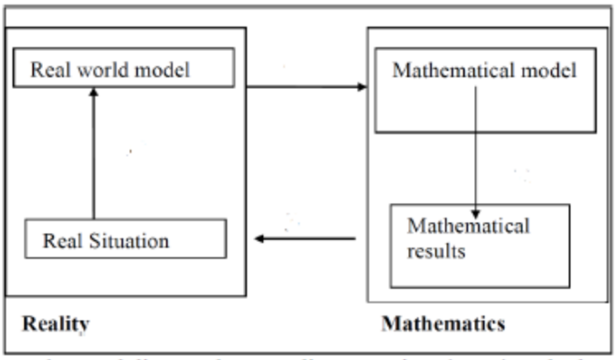
Appendix 7: Various mathematical modelling diagrams

By Mason and Davis's (1991, p. 51)

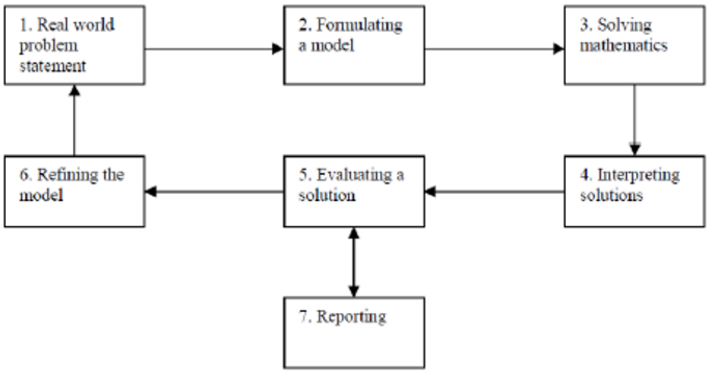


Mason and Davis's (1991) modelling process

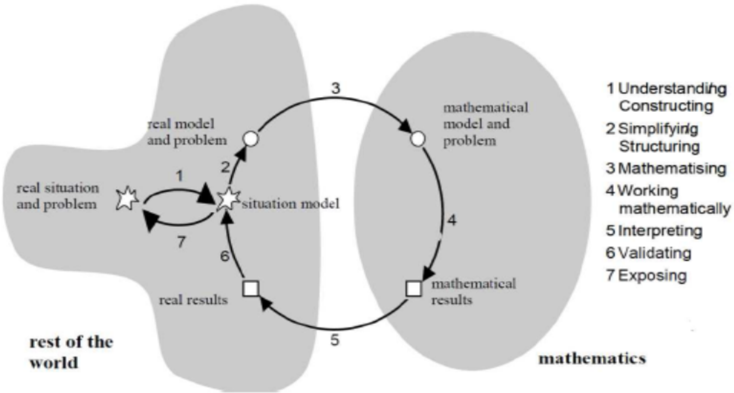
By Perrenet and Zwaneveld (2012, pp. 4-6). Literature review.



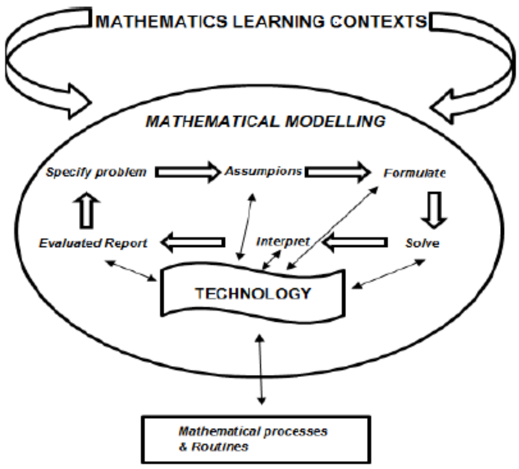
The Modeling cycle according to Kaiser (1995) and Blum (1996)



Modeling cycle according to Berry and Davies (2006)



The modeling cycle according to Blum and Leiß (2006)



Modeling cycle according to Geiger (2011)

Appendix 8: Currently Chilean mathematics curriculum

	Age of pupils at the beginning of the academic year	Academic year of implementation of the basic curriculum, including mathematics (from March)	Decree
First Year	6 years old	2012	N° 439/2012
Second Year	7 years old		
Third Year	8 years old		
Fourth Year	9 years old	2013	
Fifth Year	10 years old		
Sixth Year	11 years old		
Seventh Year	12 years old	2016	N° 614/2013
Eighth Year	13 years old	2016	
Ninth Year	14 years old	2017	Nº 169 exento, de 2014
Tenth Year	15 years old	2018	

Appendix 9: Mathematical modelling task (Spanish version)

Problema de Presupuesto

Tú perteneces a un equipo que hace y vende juegos para eventos de Kermesse. En estos momentos, están diseñando un juego con 5 botellas de agua y 5 bolsas de porotos.

Se sabe que el precio de cada botella de agua es de $ 1 y el de cada bolsa de porotos es de $ 1,25. Además, los premios pequeños que se darán al jugar el juego cuestan $0,50 c/u, los medianos $1,00 c/u y los grandes $ 3,25 c/u.

Ustedes esperan que jueguen 175 niños y planean cobrar $250 por cada juego (incluido los premios) a los organizadores de la Kermesse. Además, desean ganar $100 por la venta del juego, por lo tanto tú y tu equipo tienen $150 para gastar en cada juego con los premios.

Planeen como gastar su presupuesto usando la matemática para mostrar que el plan de tu equipo funcionará con 175 niños jugando, para ello hagan un poster con:

- Un dibujo del juego
- Las reglas del juego
- Una hoja que muestre como gastarán su presupuesto.

- Extra. Si tuvieran que revisar su presupuesto y cambiarlo a $100 para gastar, ¿qué te gustaría cambiar? Justifica tu respuesta.

Note que todos los valores están escritos en dólares.